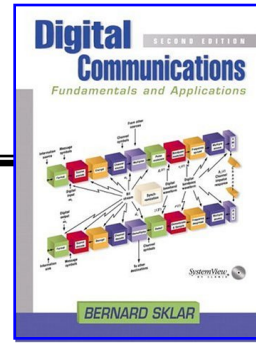

DIGITAL COMMUNICATIONS

Fundamentals and Applications

Second Edition

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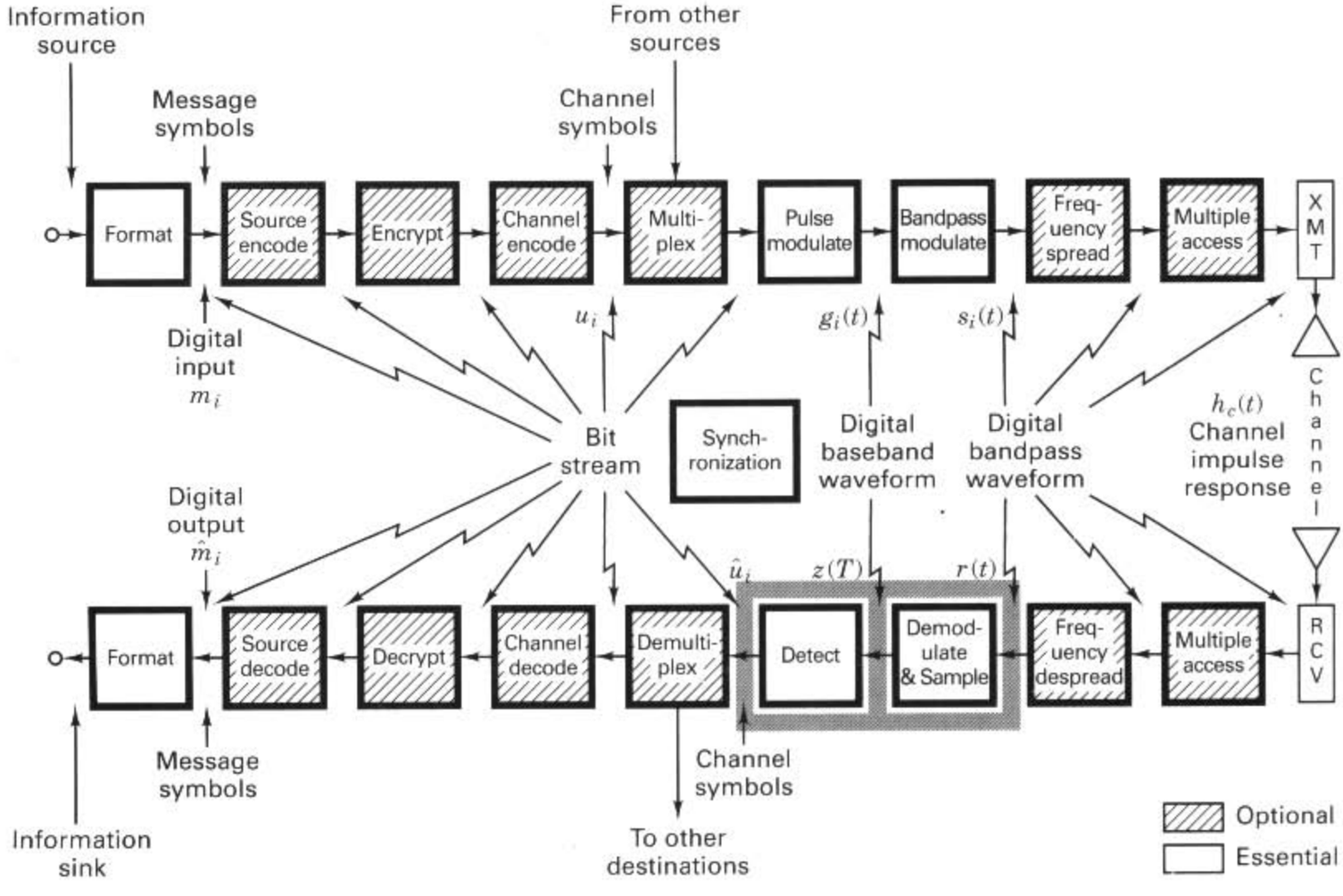
CHAPTER 3

Baseband Demodulation/Detection



In the case of baseband signaling, the received waveforms are already in a pulse-like form. One might ask, why then, is a demodulator needed to recover the pulse waveforms? The answer is that the arriving baseband pulses are not in the form of ideal pulse shapes, each one occupying its own symbol interval. The filtering at the transmitter and the channel typically cause the received pulse sequence to suffer from intersymbol interference (ISI) and thus appear as an amorphous "smeared" signal, not quite ready for sampling and detection. The goal of the demodulator (receiving filter) is to recover a baseband pulse with the best possible signal-to-noise ration (SNR), free of any ISI. Equalization, covered in this chapter, is a technique used to help accomplish this goal. The equalization process is not required for every type of communication channel. However, since equalization embodies a sophisticated set of signal-processing techniques, making it possible to compensate for channel-induced interference, it is an important area for many systems.

The bandpass model of the detection process, covered in Chapter 4, is virtually identical to the baseband model considered in this chapter. That is because a received bandpass waveform is first transformed to a baseband waveform before the final detection step takes place. For linear systems, the mathematics of detection is unaffected by a shift in frequency. In fact, we can define an *equivalence theorem* as follows: Performing bandpass linear signal processing followed by heterodyning the signal to baseband, yields the same results as heterodyning the bandpass signal to baseband, followed by baseband linear signal processing. The term "heterodyning" refers to a frequency *conversion* or *mixing* process that yields

a spectral shift in the signal. As a result of this equivalence theorem, all linear signal-processing simulations can take place at baseband (which is preferred for simplicity) with the same results as at bandpass. This means that the performance of most digital communication systems will often be described and analyzed as if the transmission channel is a baseband channel.

3.1 SIGNALS AND NOISE

3.1.1 Error-Performance Degradation in Communication Systems

The task of the detector is to retrieve the bit stream from the received waveform, as error free as possible, notwithstanding the impairments to which the signal may have been subjected. There are two primary causes for error-performance degradation. The first is the effect of filtering at the transmitter, channel, and receiver, discussed in Section 3.3, below. As described there, a nonideal system transfer function causes symbol "smearing" or *intersymbol interference* (ISI).

Another cause for error-performance degradation is electrical noise and interference produced by a variety of sources, such as galaxy and atmospheric noise, switching transients, intermodulation noise, as well as interfering signals from other sources. (These are discussed in Chapter 5.) With proper precautions, much of the noise and interference entering a receiver can be reduced in intensity or even eliminated. However, there is one noise source that cannot be eliminated, and that is the noise caused by the thermal motion of electrons in any conducting media. This motion produces *thermal noise* in amplifiers and circuits, and corrupts the signal in an additive fashion. The statistics of thermal noise have been developed using quantum mechanics, and are well known [1].

The primary statistical characteristic of thermal noise is that the noise amplitudes are distributed according to a normal or Gaussian distribution, discussed in Section 1.5.5, and shown in Figure 1.7. In this figure, it can be seen that the most probable noise amplitudes are those with small positive or negative values. In theory, the noise can be infinitely large, but very large noise amplitudes are rare. The primary spectral characteristic of thermal noise in communication systems, is that its two-sided power spectral density $G_n(f) = N_0/2$ is flat for all frequencies of interest. In other words, the thermal noise, on the average, has just as much power per hertz in low-frequency fluctuations as in high-frequency fluctuations—up to a frequency of about 10^{12} hertz. When the noise power is characterized by such a constant-power spectral density, we refer to it as *white noise.* Since thermal noise is present in all communication systems and is the predominant noise source for many systems, the thermal noise characteristics (additive, white, and Gaussian, giving rise to the name AWGN) are most often used to model the noise in the detection process and in the design of receivers. Whenever a channel is designated as an AWGN channel (with no other impairments specified), we are in effect being told that its impairments are limited to the degradation caused by this unavoidable thermal noise.

3.1.2 Demodulation and Detection

During a given signaling interval T, a binary baseband system will transmit one of two waveforms, denoted $g_1(t)$ and $g_2(t)$. Similarly, a binary bandpass system will transmit one of two waveforms, denoted $s_1(t)$ and $s_2(t)$. Since the general treatment of demodulation and detection are essentially the same for baseband and bandpass systems, we use $s_i(t)$ here as a generic designation for a transmitted waveform, whether the system is baseband or bandpass. This allows much of the baseband demodulation/detection treatment in this chapter to be consistent with similar bandpass descriptions in Chapter 4. Then, for any binary channel, the transmitted signal over a symbol interval $(0, T)$ is represented by

$$s_i(t) = \begin{cases} s_1(t) & 0 \le t \le T \quad \text{for a binary 1} \\ s_2(t) & 0 \le t \le T \quad \text{for a binary 0} \end{cases}$$

The received signal $r(t)$ degraded by noise $n(t)$ and possibly degraded by the impulse response of the channel $h_c(t)$ was described in Equation (1.1) and is rewritten as

$$r(t) = s_i(t) * h_c(t) + n(t) \qquad i = 1, \ldots, M \tag{3.1}$$

where $n(t)$ is here assumed to be a zero mean AWGN process, and $*$ represents a convolution operation. For binary transmission over an ideal distortionless channel where convolution with $h_c(t)$ produces no degradation (since for the ideal case $h_c(t)$ is an impulse function), the representation of $r(t)$ can be simplified to

$$r(t) = s_i(t) + n(t) \qquad i = 1, 2, \qquad 0 \le t \le T \tag{3.2}$$

Figure 3.1 shows the typical demodulation and detection functions of a digital receiver. Some authors use the terms "demodulation" and "detection" interchangeably. This book makes a distinction between the two. We define *demodulation* as recovery of a waveform (to an undistorted baseband pulse), and we designate *detection* to mean the decision-making process of selecting the digital meaning of that waveform. If error-correction coding *not* present, the detector output consists of estimates of message symbols (or bits), $\hat{m}_i$ (also called *hard decisions*). If error-correction coding is used, the detector output consists of estimates of channel symbols (or coded bits) $\hat{u}_i$, which can take the form of *hard* or *soft decisions* (see Section 7.3.2). For brevity, the term "detection" is occasionally used loosely to encompass all the receiver signal-processing steps through the decision making step. The *frequency down-conversion* block, shown in the demodulator portion of Figure 3.1, performs frequency translation for bandpass signals operating at some radio frequency (RF). This function may be configured in a variety of ways. It may take place within the front end of the receiver, within the demodulator, shared between the two locations, or not at all.

Within the *demodulate* and *sample* block of Figure 3.1 is the *receiving filter* (essentially the demodulator), which performs waveform recovery in preparation for the next important step—detection. The filtering at the transmitter and the channel typically cause the received pulse sequence to suffer from ISI, and thus it is

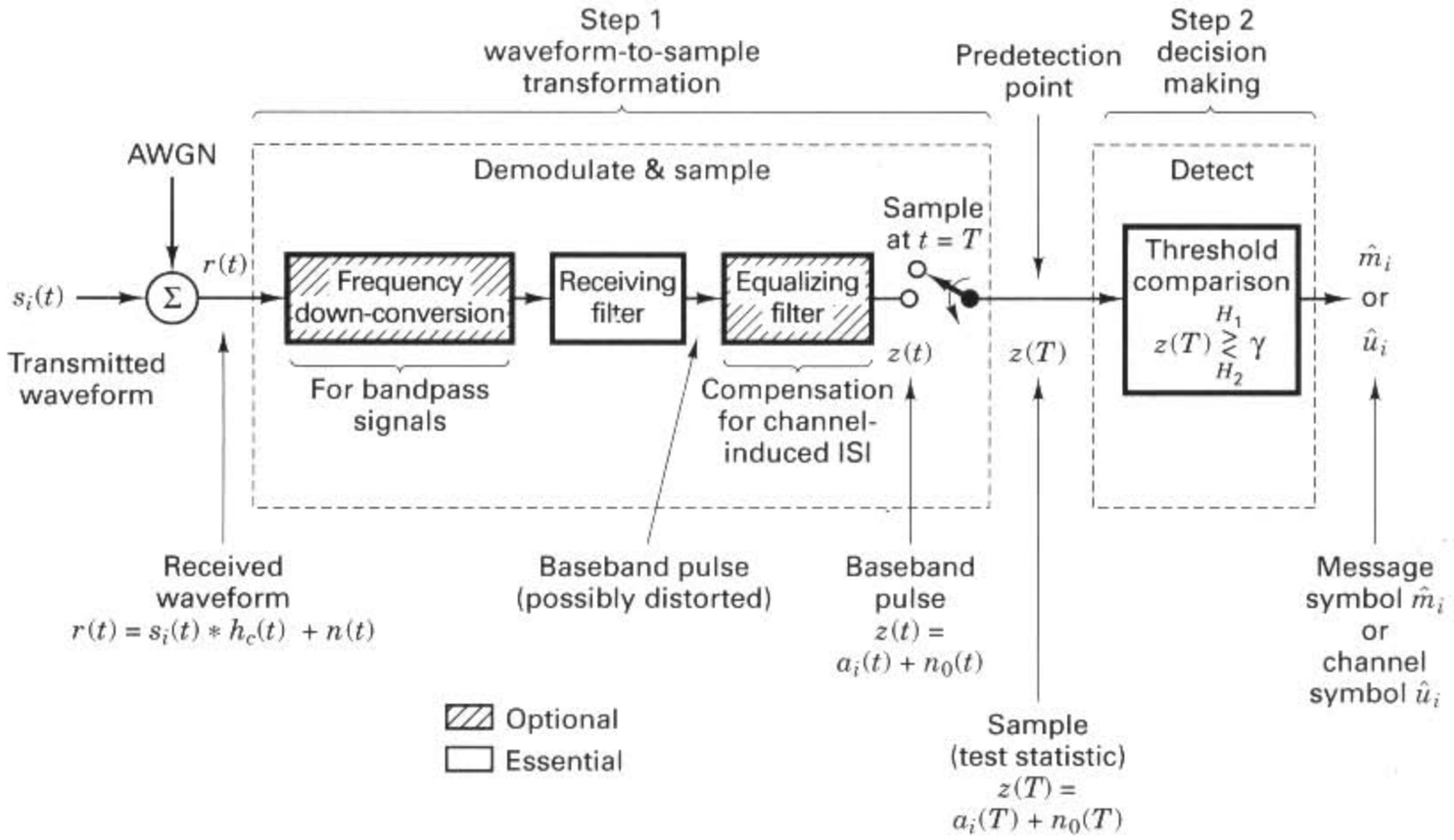


Figure 3.1 Two basic steps in the demodulation/detection of digital signals.

not quite ready for sampling and detection. The goal of the receiving filter is to recover a baseband pulse with the best possible signal-to-noise ratio (SNR), free of any ISI. The optimum receiving filter for accomplishing this is called a *matched filter* or *correlator,* described in Sections 3.2.2 and 3.2.3. An optional *equalizing filter* follows the receiving filter; it is only needed for those systems where channel-induced ISI can distort the signals. The receiving filter and equalizing filter are shown as two separate blocks in order to emphasize their separate functions. In most cases, however, when an equalizer is used, a single filter would be designed to incorporate both functions and thereby compensate for the distortion caused by both the transmitter and the channel. Such a composite filter is sometimes referred to simply as the *equalizing filter* or the *receiving and equalizing filter.*

Figure 3.1 highlights two steps in the demodulation/detection process. Step 1, the waveform-to-sample transformation, is made up of the demodulator followed by a sampler. At the end of each symbol duration T, the output of the sampler, the *predetection point,* yields a sample $z(T)$, sometimes called the test statistic. $z(T)$ has a voltage value directly proportional to the energy of the received symbol and inversely proportional to the noise. In step 2, a decision (detection) is made regarding the digital meaning of that sample. We assume that the input noise is a Gaussian random process and that the receiving filter in the demodulator is linear. A linear operation performed on a Gaussian random process will produce a second Gaussian random process [2]. Thus, the filter output noise is Gaussian. The output of step 1 yields the test statistic

$$z(T) = a_i(T) + n_0(T) \qquad i = 1, 2 \tag{3.3}$$

where $a_i(T)$ is the desired signal component, and $n_0(T)$ is the noise component. To simplify the notation, we sometimes express Equation (3.3) in the form of $z = a_i + n_0$. The noise component n_0 is a zero mean Gaussian random variable, and thus $z(T)$ is a Gaussian random variable with a mean of either a_1 or a_2 depending on whether a binary one or binary zero was sent. As described in Section 1.5.5, the probability density function (pdf) of the Gaussian random noise n_0 can be expressed as

$$p(n_0) = \frac{1}{\sigma_0 \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{n_0}{\sigma_0}\right)^2\right] \tag{3.4}$$

where σ_0^2 is the noise variance. Thus it follows from Equations (3.3) and (3.4) that the conditional pdfs $p(z|s_1)$ and $p(z|s_2)$ can be expressed as

$$p(z|s_1) = \frac{1}{\sigma_0 \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z - a_1}{\sigma_0}\right)^2\right] \tag{3.5}$$

and

$$p(z|s_2) = \frac{1}{\sigma_0 \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z - a_2}{\sigma_0}\right)^2\right] \tag{3.6}$$

These conditional pdfs are illustrated in Figure 3.2. The rightmost conditional pdf, $p(z|s_1)$, called the *likelihood* of s_1, illustrates the probability density function of the random variable $z(T)$, given that symbol s_1 was transmitted. Similarly, the leftmost conditional pdf, $p(z|s_2)$, called the *likelihood* of s_2, illustrates the pdf of $z(T)$, given that symbol s_2 was transmitted. The abscissa, $z(T)$, represents the full range of possible sample output values from step 1 of Figure 3.1.

After a received waveform has been transformed to a sample, the actual shape of the waveform is no longer important; all waveform types that are transformed to the same value of $z(T)$ are identical for detection purposes. Later it is shown that an optimum receiving filter (matched filter) in step 1 of Figure 3.1 maps all signals of equal energy into the same point $z(T)$. Therefore, the received *signal energy* (not its shape) is the important parameter in the detection process. This is why the detection analysis for baseband signals is the same as that for bandpass sig-

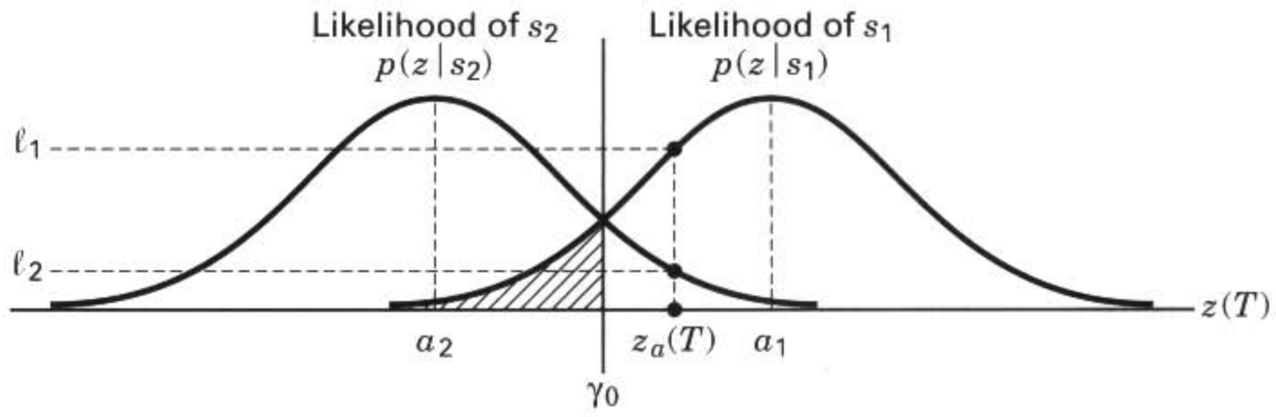


Figure 3.2 Conditional probability density functions: $p(z|s_1)$ and $p(z|s_2)$.

nals. Since $z(T)$ is a voltage signal that is proportional to the energy of the received symbol, the larger the magnitude of $z(T)$, the more error free will be the decision-making process. In step 2, detection is performed by choosing the hypothesis that results from the threshold measurement

$$z(T) \underset{H_2}{\overset{H_1}{\gtrless}} \gamma \tag{3.7}$$

where H_1 and H_2 are the two possible (binary) hypotheses. The inequality relationship indicates that hypothesis H_1 is chosen if $z(T) > \gamma$, and hypothesis H_2 is chosen if $z(T) < \gamma$. If $z(T) = \gamma$, the decision can be an arbitrary one. Choosing H_1 is equivalent to deciding that signal $s_1(t)$ was sent and hence a binary 1 is detected. Similarly, choosing H_2 is equivalent to deciding that signal $s_2(t)$ was sent, and hence a binary 0 is detected.

3.1.3 A Vector View of Signals and Noise

We now present a geometric or vector view of signal waveforms that are useful for either baseband or bandpass signals. We define an N-dimensional *orthogonal space* as a space characterized by a set of N linearly independent functions $\{\phi_j(t)\}$, called *basis functions.* Any arbitrary function in the space can be generated by a linear combination of these basis functions. The basis functions must satisfy the conditions

$$\int_0^T \psi_j(t)\psi_k(t)\,dt = K_j\delta_{jk} \qquad 0 \le t \le T \qquad j, k = 1, \ldots, N \tag{3.8a}$$

where the operator

$$\delta_{jk} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{otherwise} \end{cases} \tag{3.8b}$$

is called the *Kronecker delta function* and is defined by Equation (3.8b). When the K_j constants are nonzero, the signal space is called *orthogonal.* When the basis functions are normalized so that each $K_j = 1$, the space is called an *orthonormal* space. The principal requirement for orthogonality can be stated as follows. Each $\psi_j(t)$ function of the set of basis functions must be independent of the other members of the set. Each $\psi_j(t)$ must not interfere with any other members of the set in the detection process. From a geometric point of view, each $\psi_j(t)$ is mutually perpendicular to each of the other $\psi_k(t)$ for $j \neq k$. An example of such a space with $N = 3$ is shown in Figure 3.3, where the mutually perpendicular axes are designated $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$. If $\psi_j(t)$ corresponds to a real-valued voltage or current waveform component, associated with a 1-Ω resistive load, then using Equations (1.5) and (3.8), the normalized energy in joules dissipated in the load in T seconds, due to ψ_j, is

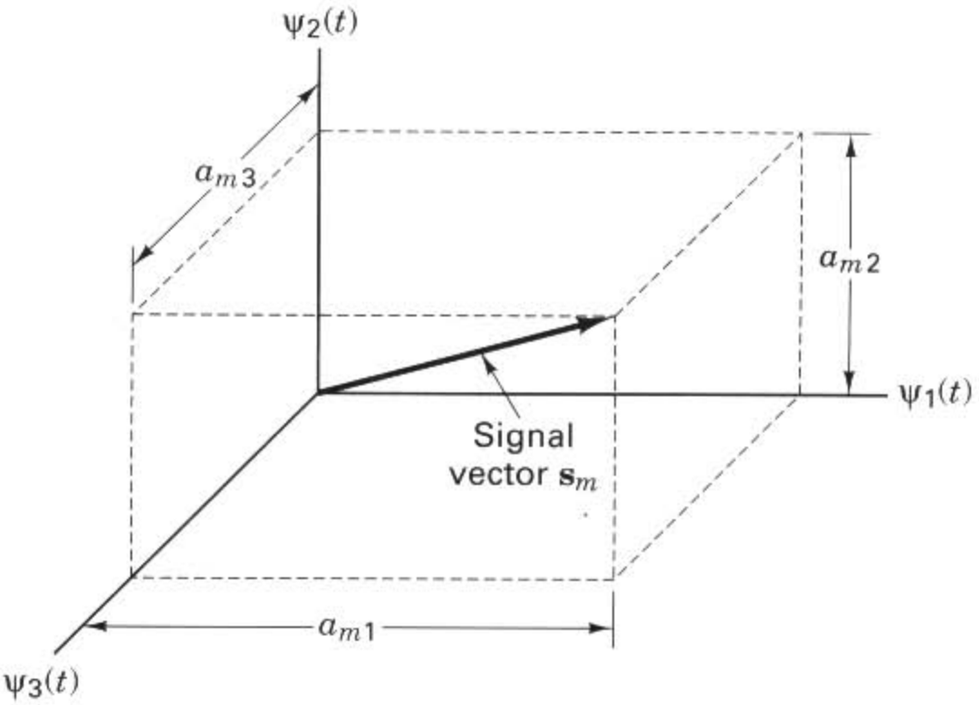


Figure 3.3 Vectorial representation of the signal waveform $s_m(t)$.

$$E_j = \int_0^T \psi_j^2(t)\, dt = K_j \tag{3.9}$$

One reason we focus on an *orthogonal signal space* is that Euclidean distance measurements, fundamental to the detection process, are easily formulated in such a space. However, even if the signaling waveforms do not make up such an orthogonal set, they can be transformed into linear combinations of orthogonal waveforms. It can be shown [3] that *any arbitrary* finite set of waveforms $\{s_i(t)\}$ $(i = 1, \ldots, M)$, where each member of the set is physically realizable and of duration T, can be expressed as a linear combination of N orthogonal waveforms $\psi_1(t)$, $\psi_2(t), \ldots, \psi_N(t)$, where $N \leq M$, such that

$$\begin{aligned}
s_1(t) &= a_{11}\psi_1(t) + a_{12}\psi_2(t) + \cdots + a_{1N}\psi_N(t) \\
s_2(t) &= a_{21}\psi_1(t) + a_{22}\psi_2(t) + \cdots + a_{2N}\psi_N(t) \\
&\vdots \qquad\qquad\qquad\qquad\qquad\quad \vdots \\
s_M(t) &= a_{M1}\psi_1(t) + a_{M2}\psi_2(t) + \cdots + a_{MN}\psi_N(t)
\end{aligned}$$

These relationships are expressed in more compact notation as

$$s_i(t) = \sum_{j=1}^{N} a_{ij}\, \psi_j(t) \quad \begin{matrix} i = 1, \ldots, M \\ N \leq M \end{matrix} \tag{3.10}$$

where

$$a_{ij} = \frac{1}{K_j} \int_0^T s_i(t)\psi_j(t)\, dt \quad \begin{matrix} i = 1, \ldots, M \quad 0 \leq t \leq T \\ j = 1, \ldots, N \end{matrix} \tag{3.11}$$

The coefficient a_{ij} is the value of the $\psi_j(t)$ component of signal $s_i(t)$. The form of the $\{\psi_j(t)\}$ is not specified; it is chosen for convenience and will depend on the form of the signal waveforms. The set of signal waveforms, $\{s_i(t)\}$, can be viewed as a set of

vectors, $\{\mathbf{s}_i\} = \{a_{i1}, a_{i2}, \ldots, a_{iN}\}$. If, for example, $N = 3$, we may plot the vector $\mathbf{s}_m$ corresponding to the waveform

$$s_m(t) = a_{m1}\psi_1(t) + a_{m2}\psi_2(t) + a_{m3}\psi_3(t)$$

as a point in a three-dimensional Euclidean space with coordinates (a_{m1}, a_{m2}, a_{m3}), as shown in Figure 3.3. The orientation among the signal vectors describes the relation of the signals to one another (with respect to phase or frequency), and the amplitude of each vector in the set $\{\mathbf{s}_i\}$ is a measure of the signal energy transmitted during a symbol duration. In general, once a set of N orthogonal functions has been adopted, each of the transmitted signal waveforms, $s_i(t)$, is completely determined by the vector of its coefficients,

$$\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN}) \qquad i = 1, \ldots, M \tag{3.12}$$

We shall employ the notation of signal vectors, $\{\mathbf{s}\}$, or signal waveforms, $\{s(t)\}$, as best suits the discussion. A typical detection problem, conveniently viewed in terms of signal vectors, is illustrated in Figure 3.4. Vectors $\mathbf{s}_j$ *and* $\mathbf{s}_k$ represent *prototype* or *reference signals* belonging to the set of M waveforms, $\{s_i(t)\}$. The receiver knows, a priori, the location in the signal space of each prototype vector belonging to the M-ary set. During the transmission of any signal, the signal is perturbed by noise so that the resultant vector that is actually received is a perturbed version (e.g., $\mathbf{s}_j + \mathbf{n}$ or $\mathbf{s}_k + \mathbf{n}$) of the original one, where $\mathbf{n}$ represents a noise vector. The noise is additive and has a Gaussian distribution; therefore, the resulting distribution of possible received signals is a cluster or cloud of points around $\mathbf{s}_j$ and $\mathbf{s}_k$. The cluster is dense in the center and becomes sparse with increasing distance from the prototype. The arrow marked "$\mathbf{r}$" represents a signal vector that might arrive at the receiver during some symbol interval. The task of the receiver is to decide whether $\mathbf{r}$ has a close "resem-

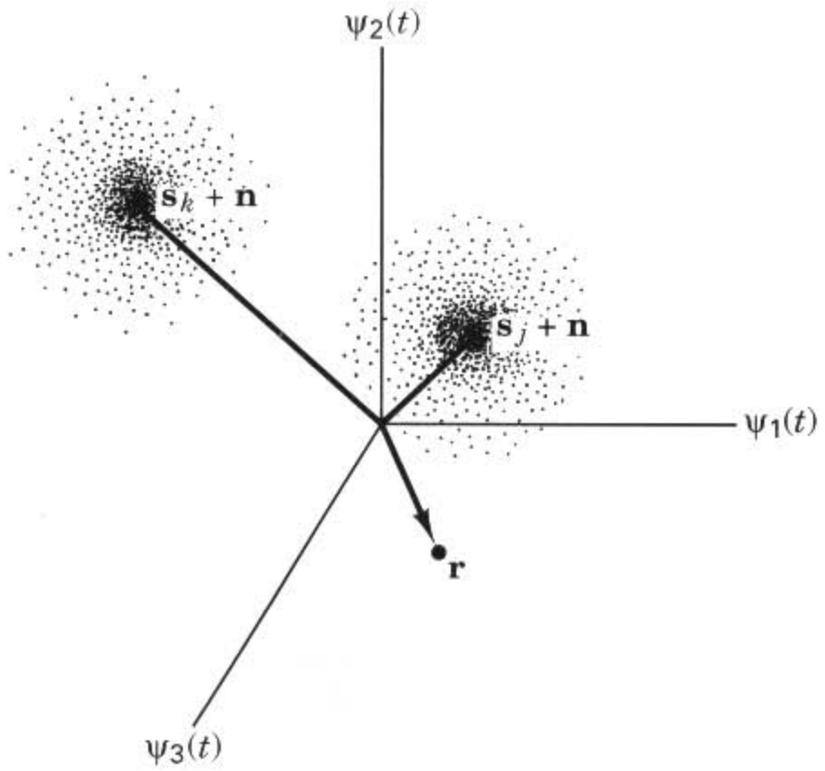


Figure 3.4 Signals and noise in a three-dimensional vector space.

blance" to the prototype $\mathbf{s}_j$, whether it more closely resembles $\mathbf{s}_k$, or whether it is closer to some other prototype signal in the M-ary set. The measurement can be thought of as a *distance* measurement. The receiver or detector must decide which of the prototypes within the signal space is *closest* in distance to the received vector $\mathbf{r}$. The analysis of all demodulation or detection schemes involves this concept of *distance* between a received waveform and a set of possible transmitted waveforms. A simple rule for the detector to follow is to decide that $\mathbf{r}$ belongs to the same class as its nearest neighbor (nearest prototype vector).

3.1.3.1 Waveform Energy

Using Equations (1.5), (3.10), and (3.8), the normalized energy E_i, associated with the waveform $s_i(t)$ over a symbol interval T can be expressed in terms of the orthogonal components of $s_i(t)$ as follows:

$$E_i = \int_0^T s_i^2(t)\,dt = \int_0^T \left[\sum_j a_{ij}\,\psi_j(t)\right]^2 dt \tag{3.13}$$

$$= \int_0^T \sum_j a_{ij}\,\psi_j(t) \sum_k a_{ik}\psi_k(t)\,dt \tag{3.14}$$

$$= \sum_j \sum_k a_{ij}\,a_{ik} \int_0^T \psi_j(t)\psi_k(t)\,dt \tag{3.15}$$

$$= \sum_j \sum_k a_{ij}\,a_{ik} K_j \delta_{jk} \tag{3.16}$$

$$= \sum_{j=1}^{N} a_{ij}^2 K_j \qquad i = 1, \ldots, M \tag{3.17}$$

Equation (3.17) is a special case of Parseval's theorem relating the integral of the square of the waveform $s_i(t)$ to the sum of the square of the orthogonal series coefficients. If orthonormal functions are used (i.e., $K_j = 1$), the normalized energy over a symbol duration T is given by

$$E_i = \sum_{j=1}^{N} a_{ij}^2 \tag{3.18}$$

If there is equal energy E in each of the $s_i(t)$ waveforms, we can write Equation (3.18) in the form

$$E = \sum_{j=1}^{N} a_{ij}^2 \qquad \text{for all } i \tag{3.19}$$

3.1.3.2 Generalized Fourier Transforms

The transformation described by Equations (3.8), (3.10), and (3.11) is referred to as the *generalized Fourier transformation.* In the case of ordinary Fourier transforms, the $\{\psi_j(t)\}$ set comprises sine and cosine harmonic functions. But in the

case of generalized Fourier transforms, the $\{\psi_j(t)\}$ set is not constrained to any specific form; it must only satisfy the orthogonality statement of Equation (3.8). *Any* arbitrary integrable waveform set, as well as noise, can be represented as a linear combination of orthogonal waveforms through such a generalized Fourier transformation [3]. Therefore, in such an orthogonal space, we are justified in using distance (Euclidean distance) as a decision criterion for the detection of *any* signal set in the presence of AWGN. The most important application of this orthogonal transformation has to do with the way in which signals are actually transmitted and received. The transmission of a nonorthogonal signal set is generally accomplished by the appropriate weighting of the orthogonal carrier components.

Example 3.1 Orthogonal Representation of Waveforms

Figure 3.5 illustrates the statement that any arbitrary integrable waveform set can be represented as a linear combination of orthogonal waveforms. Figure 3.5a shows a set of three waveforms, $s_1(t)$, $s_2(t)$, and $s_3(t)$.

(a) Demonstrate that these waveforms *do not* form an orthogonal set.
(b) Figure 3.5b shows a set of two waveforms, $\psi_1(t)$ and $\psi_2(t)$. Verify that these waveforms form an orthogonal set.
(c) Show how the nonorthogonal waveform set in part (a) can be expressed as a linear combination of the orthogonal set in part (b).
(d) Figure 3.5c illustrates another orthogonal set of two waveforms, $\psi_1'(t)$ and $\psi_2'(t)$. Show how the nonorthogonal set in Figure 3.5a can be expressed as a linear combination of the set in Figure 3.5c.

Solution

(a) $s_1(t)$, $s_2(t)$, and $s_3(t)$ are clearly not orthogonal, since they do not meet the requirements of Equation (3.8); that is, the time integrated value (over a symbol duration) of the cross-product of any two of the three waveforms is not zero. Let us verify this for $s_1(t)$ and $s_2(t)$:

$$\int_0^T s_1(t)s_2(t)\,dt = \int_0^{T/2} s_1(t)s_2(t)\,dt + \int_{T/2}^T s_1(t)s_2(t)\,dt$$

$$= \int_0^{T/2} (-1)(2)\,dt + \int_{T/2}^T (-3)(0)\,dt = -T$$

Similarly, the integral over the interval T of each of the cross-products $s_1(t)s_3(t)$ and $s_2(t)s_3(t)$ results in nonzero values. Hence, the waveform set $\{s_i(t)\}$ $(i = 1, 2, 3)$ in Figure 3.5a is not an orthogonal set.

(b) Using Equation (3.8), we verify that $\psi_1(t)$ and $\psi_2(t)$ form an orthogonal set as follows:

$$\int_0^T \psi_1(t)\psi_2(t)\,dt = \int_0^{T/2} (1)(1)\,dt + \int_{T/2}^T (-1)(1)\,dt = 0$$

(c) Using Equation (3.11) with $K_j = T$, we can express the nonorthogonal set $\{s_i(t)\}$ $(i = 1, 2, 3)$ as a linear combination of the orthogonal basis waveforms $\{\psi_j(t)\}$ $(j = 1, 2)$:

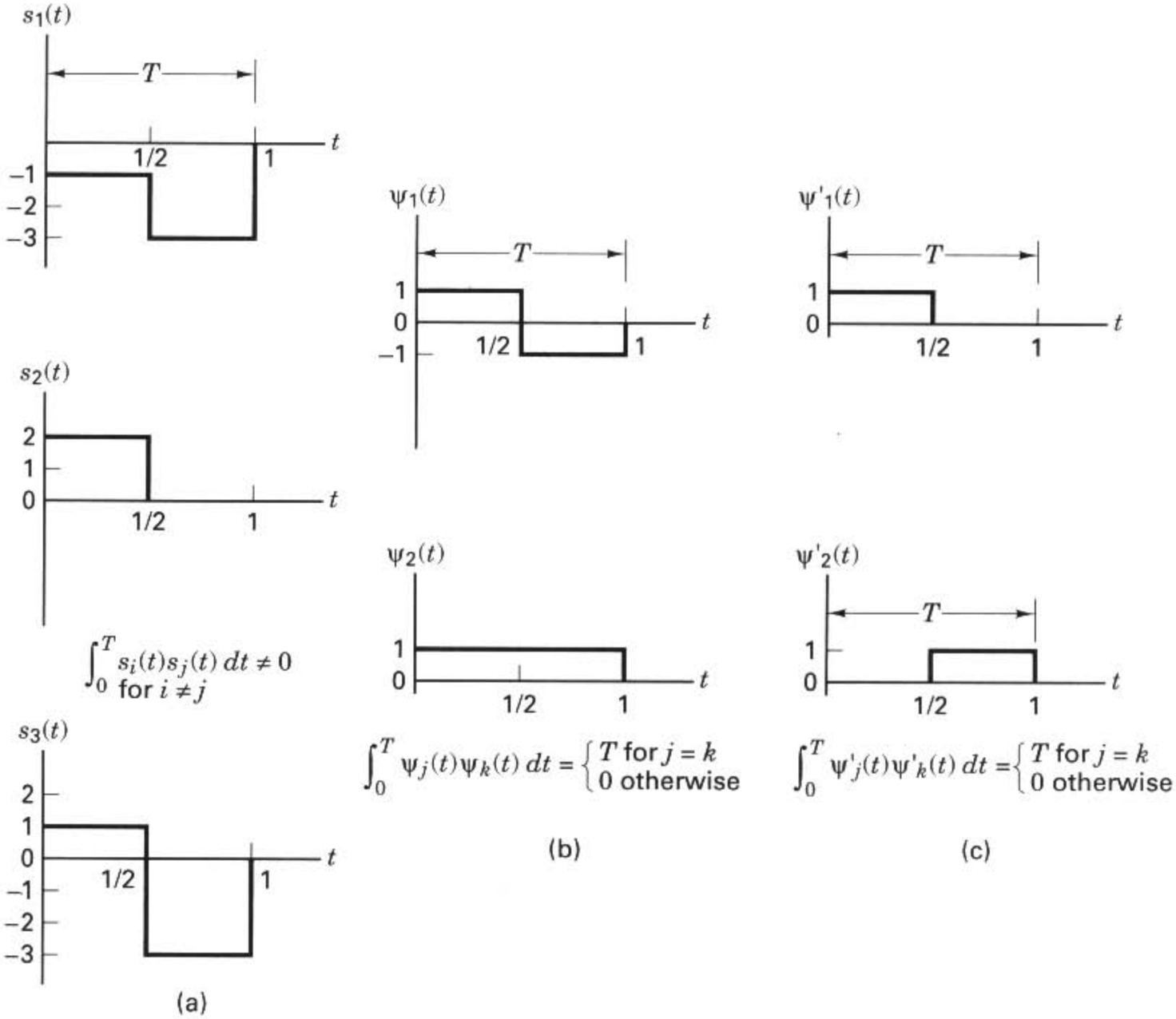


Figure 3.5 Example of an arbitrary signal set in terms of an orthogonal set. (a) Arbitrary signal set. (b) A set of orthogonal basis functions. (c) Another set of orthogonal basis functions.

$$s_1(t) = \psi_1(t) - 2\psi_2(t)$$
$$s_2(t) = \psi_1(t) + \psi_2(t)$$
$$s_3(t) = 2\psi_1(t) - \psi_2(t)$$

(d) Similar to part (c), the nonorthogonal set $\{s_i(t)\}$ $(i = 1, 2, 3)$ can be expressed in terms of the simple orthogonal basis set $\{\psi'_j(t)\}$ $(j = 1, 2)$ in Figure 3.5c, as follows:

$$s_1(t) = -\psi'_1(t) - 3\psi'_2(t)$$
$$s_2(t) = 2\psi'_1(t)$$
$$s_3(t) = \psi'_1(t) - 3\psi'_2(t)$$

These relationships illustrate how an arbitrary waveform set $\{s_i(t)\}$ can be expressed as a linear combination of an orthogonal set $\{\psi_j(t)\}$, as described in Equations (3.10) and (3.11). What are the practical applications of being able to describe $s_1(t)$, $s_2(t)$, and $s_3(t)$ in terms of $\psi_1(t)$, $\psi_2(t)$, and the appropriate coeffi-

cients? If we want a system for transmitting waveforms $s_1(t)$, $s_2(t)$, and $s_3(t)$, the transmitter and the receiver need only be implemented using the two basis functions $\psi_1(t)$ and $\psi_2(t)$ instead of the three original waveforms. The *Gram–Schmidt orthogonalization procedure* provides a convenient way in which an appropriate choice of a basis function set $\{\psi_j(t)\}$, can be obtained for any given signal set $\{s_i(t)\}$. (It is described in Appendix 4A of Reference [4].)

3.1.3.3 Representing White Noise with Orthogonal Waveforms

Additive white Gaussian noise (AWGN) can be expressed as a linear combination of orthogonal waveforms in the same way as signals. For the signal detection problem, the noise can be partitioned into two components,

$$n(t) = \hat{n}(t) + \tilde{n}(t) \tag{3.20}$$

where

$$\hat{n}(t) = \sum_{j=1}^{N} n_j \psi_j(t) \tag{3.21}$$

is taken to be the noise within the signal space, or the projection of the noise components on the signal coordinates $\psi_1(t), \ldots, \psi_N(t)$, and

$$\tilde{n}(t) = n(t) - \hat{n}(t) \tag{3.22}$$

is defined as the noise outside the signal space. In other words, $\tilde{n}(t)$ may be thought of as the noise that is effectively tuned out by the detector. The symbol $\hat{n}(t)$ represents the noise that will interfere with the detection process. We can express the noise waveform $n(t)$ as

$$n(t) = \sum_{j=1}^{N} n_j \psi_j(t) + \tilde{n}(t) \tag{3.23}$$

where

$$n_j = \frac{1}{K_j} \int_0^T n(t)\psi_j(t)\, dt \qquad \text{for all } j \tag{3.24}$$

and

$$\int_0^T \tilde{n}(t)\psi_j(t)\, dt = 0 \qquad \text{for all } j \tag{3.25}$$

The interfering portion of the noise, $\hat{n}(t)$, expressed in Equation (3.21) will henceforth be referred to simply as $n(t)$. We can express $n(t)$ by a vector of its coefficients similar to the way we did for signals in Equation (3.12). We have

$$\mathbf{n} = (n_1, n_2, \ldots, n_N) \tag{3.26}$$

where $\mathbf{n}$ is a random vector with zero mean and Gaussian distribution, and where the noise components n_i $(i = 1, \ldots, N)$ are independent.

3.1.3.4 Variance of White Noise

White noise is an *idealized process* with two-sided power spectral density equal to a constant $N_0/2$, for all frequencies from $-\infty$ to $+\infty$. Hence, the noise variance (average noise power, since the noise has zero mean) is

$$\sigma^2 = \text{var}\,[n(t)] = \int_{-\infty}^{\infty} \left(\frac{N_0}{2}\right) df = \infty \tag{3.27}$$

Although the variance for AWGN is infinite, the variance for *filtered* AWGN is finite. For example, if AWGN is correlated with one of a set of orthonormal functions $\psi_j(t)$, the variance of the correlator output is given by

$$\sigma^2 = \text{var}\,(n_j) = \mathbf{E}\left\{\left[\int_0^T n(t)\psi_j(t)\,dt\right]^2\right\} = \frac{N_0}{2} \tag{3.28}$$

The proof of Equation (3.28) is given in Appendix C. Henceforth we shall assume that the noise of interest in the detection process is the output noise of a correlator or matched filter with variance $\sigma^2 = N_0/2$ as expressed in Equation (3.28).

3.1.4 The Basic SNR Parameter for Digital Communication Systems

Anyone who has studied analog communications is familiar with the figure of merit, *average signal power to average noise power ratio* (*S/N* or SNR). In digital communications, we more often use E_b/N_0, a normalized version of SNR, as a figure of merit. E_b is bit energy and can be described as signal power S times the bit time T_b. N_0 is noise power spectral density, and can be described as noise power N divided by bandwidth W. Since the bit time and bit rate R_b are reciprocal, we can replace T_b with $1/R_b$ and write

$$\frac{E_b}{N_0} = \frac{S\,T_b}{N/W} = \frac{S/R_b}{N/W} \tag{3.29}$$

Data rate, in units of bits per second, is one of the most recurring parameters in digital communications. We therefore simplify the notation throughout the book, by using R instead of R_b to represent bits/s, and we rewrite Equation (3.29) to emphasize that E_b/N_0 is just a version of *S/N* normalized by bandwidth and bit rate, as follows:

$$\frac{E_b}{N_0} = \frac{S}{N}\left(\frac{W}{R}\right) \tag{3.30}$$

One of the most important metrics of performance in digital communication systems is a plot of the bit-error probability P_B versus E_b/N_0. Figure 3.6 illustrates the "waterfall-like" shape of most such curves. For $E_b/N_0 \geq x_0$, $P_B \leq P_0$. The dimensionless ratio E_b/N_0 is a standard quality measure for digital communications system performance. Therefore, required E_b/N_0 can be considered a metric that characterizes the performance of one system versus another; the smaller the required E_b/N_0, the more efficient is the detection process for a given probability of error.

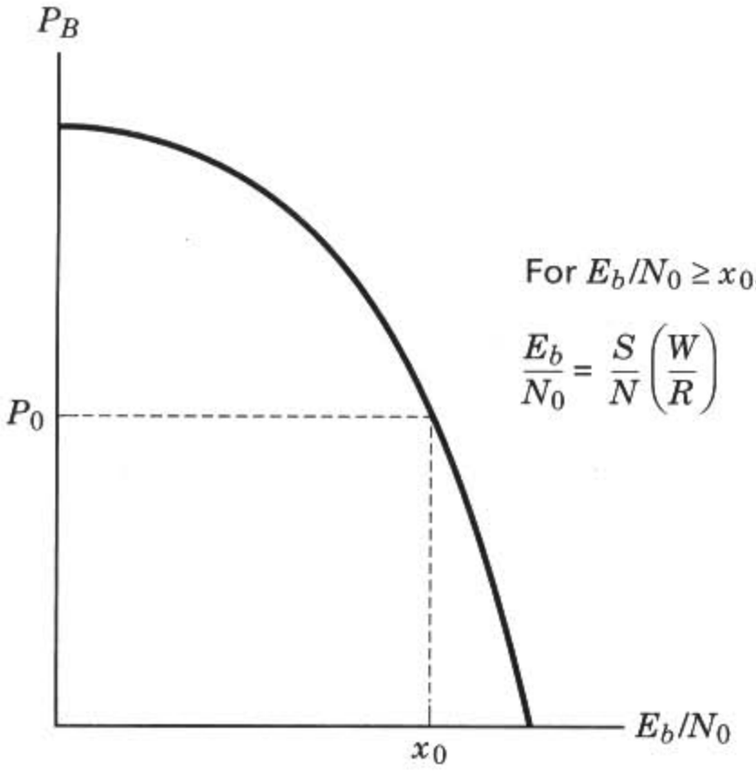


Figure 3.6 General shape of the P_B versus E_b/N_0 curve.

3.1.5 Why E_b/N_0 Is a Natural Figure of Merit

A newcomer to digital communications may question the usefulness of the parameter E_b/N_0. After all, S/N is a useful figure of merit for analog communications—the numerator represents a power measurement of the signal we wish to preserve and deliver, and the denominator represents electrical noise degradation. Moreover, S/N is intuitively acceptable as a metric of goodness. Thus, why can't we continue to use S/N as a figure of merit for digital communications? Why has a different metric for digital systems—the ratio of bit energy to noise power spectral density—arisen? The explanation is given below.

In Section 1.2.4, a power signal was defined as a signal having finite average power and infinite energy. An energy signal was defined as a signal having zero average power and finite energy. These classifications are useful in comparing analog and digital waveforms. We classify an analog waveform as a power signal. Why does this make sense? We can think of an analog waveform as having infinite duration that need not be partitioned or windowed in time. An infinitely long electrical waveform has an infinite amount of energy; hence, energy is not a useful way to characterize this waveform. Power (or rate of delivering the energy) is a more useful parameter for analog waveforms.

However, in a digital communication system we transmit (and receive) a symbol by using a transmission waveform within a window of time, the symbol time T_s. Focusing on one symbol, we can see that the power (averaged over all time) goes to zero. Hence, power is not a useful way to characterize a digital waveform. What we need for such waveforms is a metric of the "good stuff" within the window. In other words, the symbol energy (power integrated over T_s) is a more useful parameter for characterizing digital waveforms.

The fact that a digital signal is best characterized by its received energy doesn't yet get to the crux of why E_b/N_0 is a natural metric for digital systems, so let us continue. The digital waveform is a vehicle that represents a digital message. The message may contain one bit (binary), two bits (4-ary), . . . , 10 bits (1024-ary). In analog systems, there is nothing akin to such a discretized message structure. An analog information source is an infinitely quantized continuous wave. For digital systems, a figure of merit should allow us to compare one system with another at the bit level. Therefore, a description of the digital waveform in terms of *S/N* is virtually useless, since the waveform may have a one-bit meaning, a two-bit meaning, or a 10-bit meaning. For example, suppose we are told that for a given error probability, the required *S/N* for a digital binary waveform is 20 units. Think of the waveform as being interchangeable with its meaning. Since the binary waveform has a one-bit meaning, then the *S/N* requirement per bit is equal to the same 20 units. However, suppose that the waveform is 1024-ary, with the same 20 units of required *S/N*. Now, since the waveform has a 10-bit meaning, the *S/N* requirement per bit is only 2 units. Why should we have to go through such computational manipulations to find a metric that represents a figure of merit? Why not immediately describe the metric in terms of what we need—an energy-related parameter at the bit level, E_b/N_0? Just as *S/N* is a dimensionless ratio, so too is E_b/N_0. To verify this, consider the following units of measure:

$$\frac{E_b}{N_0} = \frac{\text{Joule}}{\text{Watt per Hz}} = \frac{\text{Watt-s}}{\text{Watt-s}}$$

3.2 DETECTION OF BINARY SIGNALS IN GAUSSIAN NOISE

3.2.1 Maximum Likelihood Receiver Structure

The decision-making criterion shown in step 2 of Figure 3.1 was described by Equation (3.7) as

$$z(T) \underset{H_2}{\overset{H_1}{\gtrless}} \gamma$$

A popular criterion for choosing the threshold level γ for the binary decision in Equation (3.7) is based on minimizing the probability of error. The computation for this *minimum error* value of $\gamma = \gamma_0$ starts with forming an inequality expression between the ratio of conditional probability density functions and the signal a priori probabilities. Since the conditional density function $p(z|s_i)$ is also called the *likelihood* of s_i, the formulation

$$\frac{p(z|s_1)}{p(z|s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(s_2)}{P(s_1)} \tag{3.31}$$

is called the *likelihood ratio test.* (See Appendix B.) In this inequality, $P(s_1)$ and $P(s_2)$ are the a priori probabilities that $s_1(t)$ and $s_2(t)$, respectively, are transmitted, and H_1 and H_2 are the two possible hypotheses. The rule for minimizing the error probability states that we should choose hypothesis H_1 if the ratio of likelihoods is greater than the ratio of a priori probabilities, as shown in Equation (3.31).

It is shown in Section B.3.1, that if $P(s_1) = P(s_2)$, and if the likelihoods, $p(z|s_i)$ $(i = 1, 2)$, are symmetrical, the substitution of Equations (3.5) and (3.6) into (3.31) yields

$$z(T) \underset{H_2}{\overset{H_1}{\gtrless}} \frac{a_1 + a_2}{2} = \gamma_0 \tag{3.32}$$

where a_1 is the signal component of $z(T)$ when $s_1(t)$ is transmitted, and a_2 is the signal component of $z(T)$ when $s_2(t)$ is transmitted. The threshold level γ_0, represented by $(a_1 + a_2)/2$, is the *optimum threshold* for minimizing the probability of making an incorrect decision for this important special case. This strategy is known as the *minimum error criterion.*

For equally likely signals, the optimum threshold γ_0 passes through the intersection of the likelihood functions, as shown in Figure 3.2. Thus by following Equation (3.32), the decision stage effectively selects the hypothesis that corresponds to the signal with the *maximum likelihood.* For example, given an arbitrary detector output value $z_a(T)$, for which there is a nonzero likelihood that $z_a(T)$ belongs to either signal class $s_1(t)$ or $s_2(t)$, one can think of the likelihood test as a comparison of the likelihood values $p(z_a|s_1)$ and $p(z_a|s_2)$. The signal corresponding to the maximum pdf is chosen as the most likely to have been transmitted. In other words, the detector chooses $s_1(t)$ if

$$p(z_a|s_1) > p(z_a|s_2) \tag{3.33}$$

Otherwise, the detector chooses $s_2(t)$. A detector that minimizes the error probability (for the case where the signal classes are equally likely) is also known as a *maximum likelihood detector.*

Figure 3.2 illustrates that Equation (3.33) is just a "common sense" way to make a decision when there exists statistical knowledge of the classes. Given the detector output value $z_a(T)$, we see in Figure 3.2 that $z_a(T)$ intersects the likelihood of $s_1(t)$ at a value ℓ_1, and it intersects the likelihood of $s_2(t)$ at a value ℓ_2. What is the most reasonable decision for the detector to make? For this example, choosing class $s_1(t)$, which has the greater likelihood, is the most sensible choice. If this was an M-ary instead of a binary example, there would be a total of M likelihood functions representing the M signal classes to which a received signal might belong. The maximum likelihood decision would then be to choose the class that had the greatest likelihood of all M likelihoods. (Refer to Appendix B for a review of decision theory fundamentals.)

3.2.1.1 Error Probability

For the binary decision-making depicted in Figure 3.2, there are two ways errors can occur. An error e will occur when $s_1(t)$ is sent, and channel noise results in the receiver output signal $z(t)$ being less than γ_0. The probability of such an occurrence is

$$P(e\,|s_1) = P(H_2|s_1) = \int_{-\infty}^{\gamma_0} p(z\,|s_1)\,dz \tag{3.34}$$

This is illustrated by the shaded area to the left of γ_0 in Figure 3.2. Similarly, an error occurs when $s_2(t)$ is sent, and the channel noise results in $z(T)$ being greater than γ_0. The probability of this occurrence is

$$P(e\,|s_2) = P(H_1|s_2) = \int_{\gamma_0}^{\infty} p(z\,|s_2)\,dz \tag{3.35}$$

The probability of an error is the sum of the probabilities of all the ways that an error can occur. For the binary case, we can express the probability of bit error as

$$P_B = \sum_{i=1}^{2} P(e, s_i) = \sum_{i=1}^{2} P(e\,|s_i)\,P(s_i) \tag{3.36}$$

Combining Equations (3.34) to (3.36), we can write

$$P_B = P(e\,|s_1)P(s_1) + P(e\,|s_2)P(s_2) \tag{3.37a}$$

or equivalently,

$$P_B = P(H_2|s_1)P(s_1) + P(H_1|s_2)P(s_2) \tag{3.37b}$$

That is, given that signal $s_1(t)$ was transmitted, an error results if hypothesis H_2 is chosen; or given that signal $s_2(t)$ was transmitted, an error results if hypothesis H_1 is chosen. For the case where the a priori probabilities are equal [that is, $P(s_1) = P(s_2) = \frac{1}{2}$],

$$P_B = \tfrac{1}{2}\,P(H_2|s_1) + \tfrac{1}{2}\,P(H_1|s_2) \tag{3.38}$$

and because of the symmetry of the probability density functions,

$$P_B = P(H_2|s_1) = P(H_1|s_2) \tag{3.39}$$

The probability of a bit error, P_B, is numerically equal to the area under the "tail" of either likelihood function, $p(z|s_1)$ or $p(z|s_2)$, falling on the "incorrect" side of the threshold. We can therefore compute P_B by integrating $p(z|s_1)$ between the limits $-\infty$ and γ_0, or by integrating $p(z|s_2)$ between the limits γ_0 and ∞:

$$P_B = \int_{\gamma_0 = (a_1 + a_2)/2}^{\infty} p(z\,|s_2)\,dz \tag{3.40}$$

Here, $\gamma_0 = (a_1 + a_2)/2$ is the optimum threshold from Equation (3.32). Replacing the likelihood $p(z|s_2)$ with its Gaussian equivalent from Equation (3.6), we have

$$P_B = \int_{\gamma_0=(a_1+a_2)/2}^{\infty} \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a_2}{\sigma_0}\right)^2\right] dz \tag{3.41}$$

where σ_0^2 is the variance of the noise out of the correlator.

Let $u = (z - a_2)/\sigma_0$. Then $\sigma_0\, du = dz$ and

$$P_B = \int_{u=(a_1-a_2)/2\sigma_0}^{u=\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du = Q\left(\frac{a_1-a_2}{2\sigma_0}\right) \tag{3.42}$$

where $Q(x)$, called the *complementary error function* or *co-error function*, is a commonly used symbol for the probability under the tail of the Gaussian pdf. It is defined as

$$Q(x) \approx \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{u^2}{2}\right) du \tag{3.43}$$

Note that the co-error function is defined in several ways (see Appendix B); however, all definitions are equally useful for determining probability of error in Gaussian noise. $Q(x)$ cannot be evaluated in closed form. It is presented in tabular form in Table B.1. Good approximations to $Q(x)$ by simpler functions can be found in Reference [5]. One such approximation, valid for $x > 3$, is

$$Q(x) \approx \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{3.44}$$

We have optimized (in the sense of minimizing P_B) the threshold level γ, but have not optimized the receiving filter in block 1 of Figure 3.1. We next consider optimizing this filter by maximizing the argument of $Q(x)$ in Equation (3.42).

3.2.2 The Matched Filter

A matched filter is a linear filter designed to provide the maximum signal-to-noise power ratio at its output for a given transmitted symbol waveform. Consider that a known signal $s(t)$ plus AWGN $n(t)$ is the input to a linear, time-invariant (receiving) filter followed by a sampler, as shown in Figure 3.1. At time $t = T$, the sampler output $z(T)$ consists of a signal component a_i and a noise component n_0. The variance of the output noise (average noise power) is denoted by σ_0^2, so that the ratio of the instantaneous signal power to average noise power, $(S/N)_T$, at time $t = T$, out of the sampler in step 1, is

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} \tag{3.45}$$

We wish to find the filter transfer function $H_0(f)$ that *maximizes* Equation (3.45). We can express the signal $a_i(t)$ at the filter output in terms of the filter transfer function $H(f)$ (before optimization) and the Fourier transform of the input signal, as

$$a_i(t) = \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi ft}\,df \tag{3.46}$$

where $S(f)$ is the Fourier transform of the input signal, $s(t)$. If the two-sided power spectral density of the input noise is $N_0/2$ watts/hertz, then, using Equations (1.19) and (1.53), we can express the output noise power as

$$\sigma_0^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df \tag{3.47}$$

We then combine Equations (3.45) to (3.47) to express $(S/N)_T$, as follows:

$$\left(\frac{S}{N}\right)_T = \frac{\left|\int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT}\,df\right|^2}{N_0/2\int_{-\infty}^{\infty} |H(f)|^2\,df} \tag{3.48}$$

We next find that value of $H(f) = H_0(f)$ for which the maximum $(S/N)_T$ is achieved, by using *Schwarz's inequality.* One form of the inequality can be stated as

$$\left|\int_{-\infty}^{\infty} f_1(x)f_2(x)\,dx\right|^2 \le \int_{-\infty}^{\infty} |f_1(x)|^2\,dx \int_{-\infty}^{\infty} |f_2(x)|^2\,dx \tag{3.49}$$

The equality holds if $f_1(x) = kf_2^*(x)$, where k is an arbitrary constant and * indicates complex conjugate. If we identify $H(f)$ with $f_1(x)$ and $S(f)\,e^{j2\pi fT}$ with $f_2(x)$, we can write

$$\left|\int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT}df\right|^2 \le \int_{-\infty}^{\infty} |H(f)|^2\,df \int_{-\infty}^{\infty} |S(f)|^2\,df \tag{3.50}$$

Substituting into Equation (3.48) yields

$$\left(\frac{S}{N}\right)_T \le \frac{2}{N_0}\int_{-\infty}^{\infty} |S(f)|^2\,df \tag{3.51}$$

or

$$\max\left(\frac{S}{N}\right)_T = \frac{2E}{N_0} \tag{3.52}$$

where the energy E of the input signal $s(t)$ is

$$E = \int_{-\infty}^{\infty} |S(f)|^2\,df \tag{3.53}$$

Thus, the maximum output $(S/N)_T$ depends on the input *signal energy* and the power spectral density of the noise, *not on the particular shape* of the waveform that is used.

The equality in Equation (3.52) holds only if the optimum filter transfer function $H_0(f)$ is employed, such that

$$H(f) = H_0(f) = kS^*(f)e^{-j2\pi fT} \tag{3.54}$$

or

$$h(t) = \mathcal{F}^{-1}\{kS^*(f)e^{-j2\pi fT}\} \tag{3.55}$$

Since $s(t)$ is a real-valued signal, we can write, from Equations (A.29) and (A.31),

$$h(t) = \begin{cases} ks(T-t) & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \tag{3.56}$$

Thus, the impulse response of a filter that produces the maximum output signal-to-noise ratio is the mirror image of the message signal $s(t)$, *delayed* by the symbol time duration T. Note that the delay of T seconds makes Equation (3.56) *causal;* that is, the delay of T seconds makes $h(t)$ a function of positive time in the interval $0 \le t \le T$. Without the delay of T seconds, the response $s(-t)$ is unrealizable because it describes a response as a function of negative time.

3.2.3 Correlation Realization of the Matched Filter

Equation (3.56) and Figure 3.7a illustrate the matched filter's basic property: The impulse response of the filter is a delayed version of the mirror image (rotated on the $t = 0$ axis) of the signal waveform. Therefore, if the signal waveform is $s(t)$, its mirror image is $s(-t)$, and the mirror image delayed by T seconds is $s(T - t)$. The output $z(t)$ of a causal filter can be described in the time domain as the convolution of a received input waveform $r(t)$ with the impulse response of the filter (see Section A.5):

$$z(t) = r(t) * h(t) = \int_0^t r(\tau)h(t-\tau)\,d\tau \tag{3.57}$$

Substituting $h(t)$ of Equation (3.56) into $h(t-\tau)$ of Equation (3.57) and arbitrarily setting the constant k equal to unity, we get

$$\begin{aligned} z(t) &= \int_0^t r(\tau)s[T-(t-\tau)]\,d\tau \\ &= \int_0^t r(\tau)s(T-t+\tau)\,d\tau \end{aligned} \tag{3.58}$$

When $t = T$, we can write Equation (3.58) as

$$z(T) = \int_0^T r(\tau)s(\tau)\,d\tau \tag{3.59}$$

The operation of Equation (3.59), the product integration of the received signal $r(t)$ with a replica of the transmitted waveform $s(t)$ over one symbol interval is known as the *correlation* of $r(t)$ with $s(t)$. Consider that a received signal $r(t)$ is correlated with each prototype signal $s_i(t)$ $(i = 1, \ldots, M)$, using a bank of M correlators. The signal $s_i(t)$ whose product integration or correlation with $r(t)$ yields the maximum

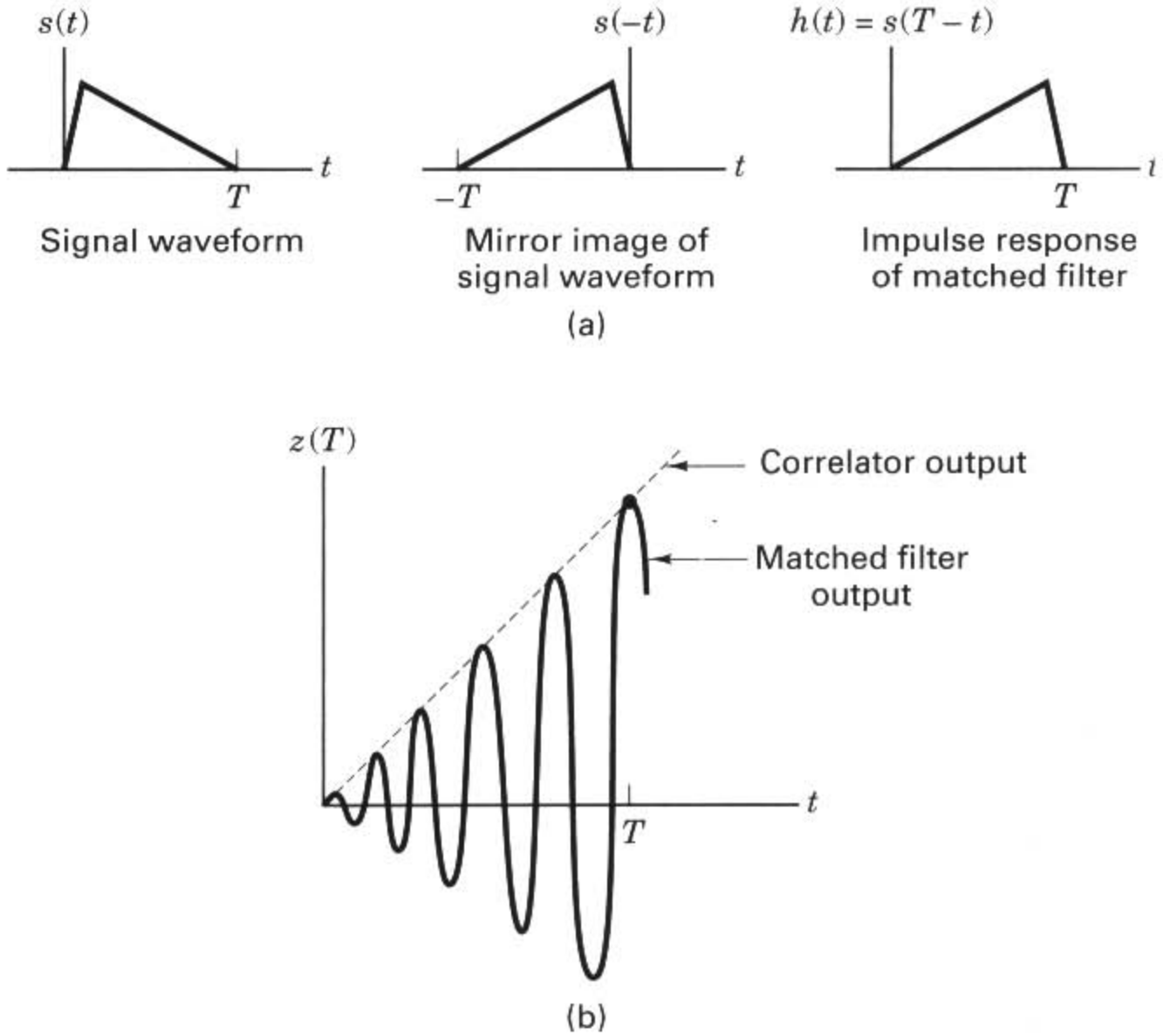


Figure 3.7 Correlator and matched filter. (a) Matched filter characteristic. (b) Comparison of correlator and matched filter outputs.

output $z_i(T)$ is the signal that matches $r(t)$ better than all the other $s_j(t)$, $j \neq i$. We will subsequently use this correlation characteristic for the optimum detection of signals.

3.2.3.1 Comparison of Convolution and Correlation

The mathematical operation of a matched filter (MF) is *convolution;* a signal is convolved with the impulse response of a filter. The mathematical operation of a correlator is *correlation;* a signal is correlated with a replica of itself. The term "matched filter" is often used synonymously with "correlator." How is that possible when their mathematical operations are different? Recall that the process of convolving two signals reverses one of them in time. Also, the impulse response of an MF is defined in terms of a signal that is reversed in time. Thus, convolution in the MF with a time-reversed function results in a second time-reversal, making the output (at the end of a symbol time) appear to be the result of a signal that has been correlated with its replica. Therefore, it is valid to implement the receiving filter in Figure 3.1 with either a matched filter or a correlator. It is important to note that the correlator output and the matched filter output are the same *only at time*

$t = T$. For a sine-wave input, the output of the correlator, $z(t)$, is approximately a linear ramp during the interval $0 \le t \le T$. However, the matched filter output is approximately a sine wave that is amplitude modulated by a linear ramp during the same time interval. The comparison is shown in Figure 3.7b. Since for comparable imputs, the MF output and the correlator output are identical at the sampling time $t = T$, the matched filter and correlator functions pictured in Figure 3.8 are often used interchangeably.

3.2.3.2 Dilemma in Representing Earliest versus Latest Event

A serious dilemma exists in representing timed events. This dilemma is undoubtedly the cause of a frequently made error in electrical engineering—confusing the most significant bit (MSB) with the least significant bit (LSB). Figure 3.9a illustrates how a function of time is typically plotted; the earliest event appears leftmost, and the latest event rightmost. In western societies, where we read from left to right, would there be any other way to plot timed events? Consider Figure 3.9b, where pulses are shown entering (and leaving) a network or circuit. Here, the earliest events are shown rightmost, and the latest are leftmost. From the figure, it should be clear that whenever we denote timed events, there is an inference that we are following one of the two formats described here. Often, we have to provide some descriptive words (e.g., the rightmost bit is the earliest bit) to avoid confusion.

Mathematical relationships often have built-in features guaranteeing the proper alignment of time events. For example, in Section 3.2.3, a matched filter is defined as having an impulse response $h(t)$ that is a delayed version of the time-reversed copy of the signal. That is, $h(t) = s(T - t)$. Delay of one symbol time T is needed for the filter to be causal (the output must occur in positive time). Time reversal can be thought of as a "precorrection" where the rightmost part of the time plot will now correspond to the earliest event. Since convolution dictates another time reversal, the arriving signal and the filter's impulse response will be "in step" (earliest with earliest, and latest with latest).

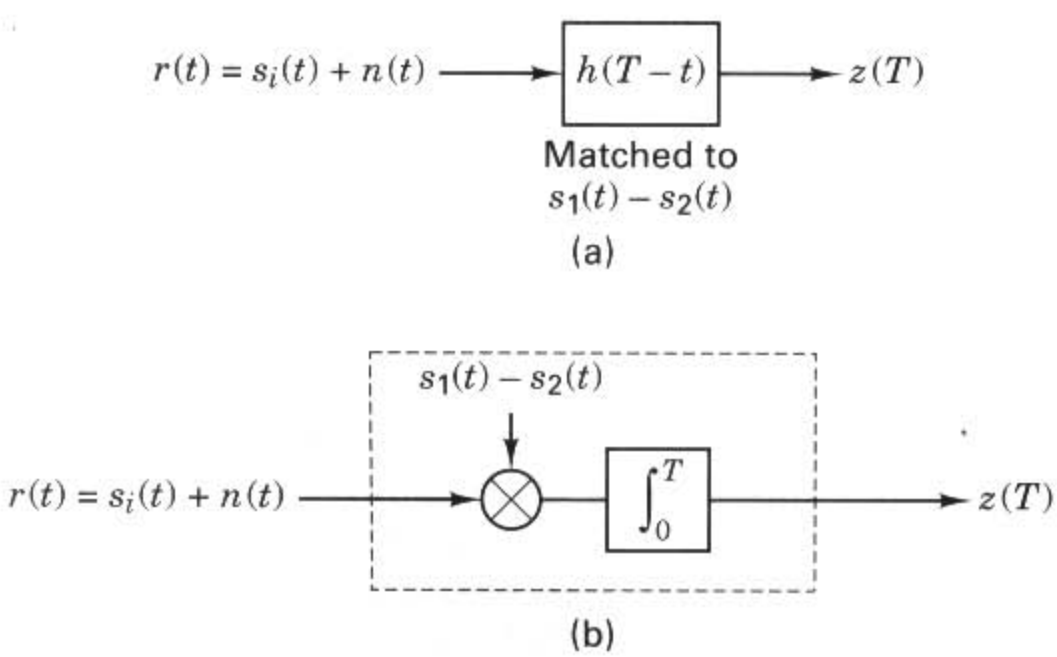


Figure 3.8 Equivalence of matched filter and correlator. (a) Matched filter. (b) Correlator.

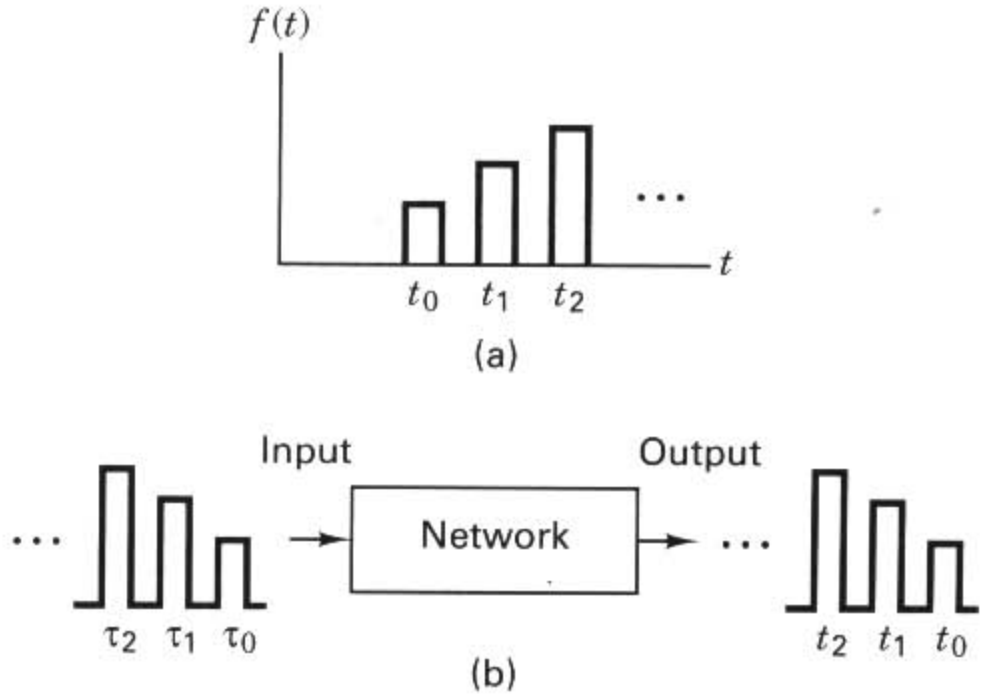


Figure 3.9 Dilemma in representing earliest versus latest events.

3.2.4 Optimizing Error Performance

To optimize (minimize) P_B in the context of an AWGN channel and the receiver shown in Figure 3.1, we need to select the optimum receiving filter in step 1 and the optimum decision threshold in step 2. For the binary case, the optimum decision threshold has already been chosen in Equation (3.32), and it was shown in Equation (3.42) that this threshold results in $P_B = Q\,[(a_1 - a_2)/2\sigma_0]$. Next, for minimizing P_B, it is necessary to choose the filter (matched filter) that maximizes the argument of $Q(x)$. Thus, we need to determine the linear filter that maximizes $(a_1 - a_2)/2\sigma_0$, or equivalently, that maximizes

$$\frac{(a_1 - a_2)^2}{\sigma_0^2} \tag{3.60}$$

where $(a_1 - a_2)$ is the difference of the desired signal components at the filter output at time $t = T$, and the square of this difference signal is the instantaneous power of the difference signal. In Section 3.2.2, a matched filter was described as one that maximizes the output signal-to-noise ratio (SNR) for a given known signal. Here, we continue that development for binary signaling, where we view the optimum filter as one that maximizes the difference between two possible signal outputs. Starting with Equations (3.45) and (3.47), it was shown in Equation (3.52) that a matched filter achieves the maximum possible output SNR equal to $2E/N_0$. Consider that the filter is matched to the input difference signal $[s_1(t) - s_2(t)]$; thus, we can write an output SNR at time $t = T$ as

$$\left(\frac{S}{N}\right)_T = \frac{(a_1 - a_2)^2}{\sigma_0^2} = \frac{2E_d}{N_0} \tag{3.61}$$

where $N_0/2$ is the two-sided power spectral density of the noise at the filter input and

$$E_d = \int_0^T [s_1(t) - s_2(t)]^2 \, dt \tag{3.62}$$

is the energy of the difference signal at the filter input. Note that Equation (3.61) does not represent the SNR for any single transmission, $s_1(t)$ or $s_2(t)$. This SNR yields a metric of signal difference for the filter output. By maximizing the output SNR as shown in Equation (3.61), the matched filter provides the maximum distance (normalized by noise) between the two candidate outputs—signal a_1 and signal a_2.

Next, combining Equations (3.42) and (3.61) yields

$$P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right). \tag{3.63}$$

For the matched filter, Equation (3.63) is an important interim result in terms of the energy of the difference signal at the filter's input. From this equation, a more general relationship in terms of received bit energy can be developed. We start by defining a time cross-correlation coefficient ρ as a measure of similarity between two signals, $s_1(t)$ and $s_2(t)$. We have

$$\rho = \frac{1}{E_b} \int_0^T s_1(t)\, s_2(t) \, dt \tag{3.64a}$$

and

$$\rho = \cos \theta \tag{3.64b}$$

where $-1 \leq \rho \leq 1$. Equation (3.64a) is the classical mathematical way of expressing correlation. However, when $s_1(t)$ and $s_2(t)$ are viewed as signal vectors, $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively, then ρ is conveniently expressed by Equation (3.64b). This vector view provides a useful image. The vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ are separated by the angle θ; for small angular separation, the vectors are quite similar (highly correlated) to each other, and for large angular separation, they are quite dissimilar. The cosine of this angle gives us the same normalized metric of correlation as Equation (3.64a).

Expanding Equation (3.62), we get

$$E_d = \int_0^T s_1^2(t) \, dt + \int_0^T s_2^2(t) \, dt - 2 \int_0^T s_1(t)\, s_2(t) \, dt \tag{3.65}$$

Recall that each of the first two terms in Equation (3.65) represents the energy associated with a bit, E_b; that is,

$$E_b = \int_0^T s_1^2(t) \, dt = \int_0^T s_2^2(t) \, dt \tag{3.66}$$

Substituting Equations (3.64a) and (3.66) into Equation (3.65), we get

$$E_d = E_b + E_b - 2\rho E_b = 2E_b(1 - \rho) \tag{3.67}$$

Substituting Equation (3.67) into (3.63), we obtain

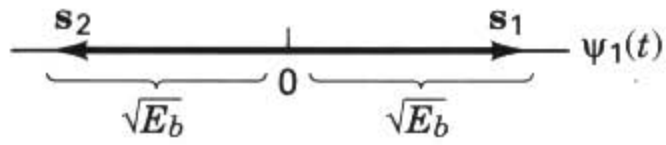
$$P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right) \tag{3.68}$$

Consider the case of $\rho = 1$ corresponding to signals $s_1(t)$ and $s_2(t)$ being perfectly correlated over a symbol time (drawn as vectors, with the angle between them equal to zero). Would anyone use such waveforms for digital signaling? Of course not, because the communication signals (members of the alphabet set) need to be as disparate from one another as possible so that they are easily distinguished (detected). We are simply cataloging the possible values for the parameter ρ. Consider the case of $\rho = -1$ corresponding to $s_1(t)$ and $s_2(t)$ being "anticorrelated" over a symbol time. In other words, the angle between the signal vectors is 180°. In such a case, where the vectors are mirror images, we call the signals *antipodal,* as shown in Figure 3.10a. Also, consider the case of $\rho = 0$ corresponding to zero correlation between $s_1(t)$ and $s_2(t)$ (the angle between the vectors is 90°). We call such signals *orthogonal,* as seen in Figure 3.10b. For two waveforms to be orthogonal, they must be uncorrelated over a symbol interval; that is,

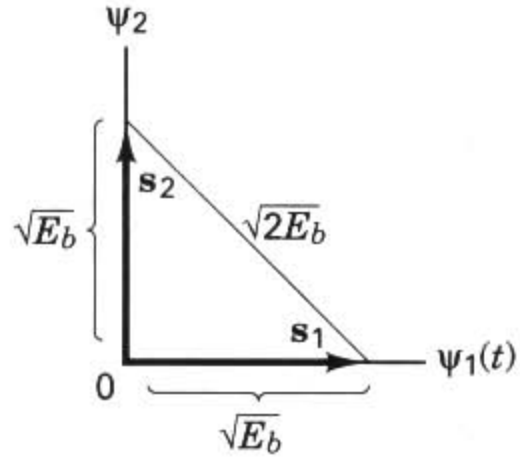
$$\int_0^T s_1(t)\, s_2(t)\, dt = 0 \tag{3.69}$$

The subject of orthogonality was treated earlier in Section 3.1.3. For the case of detecting antipodal signals (that is, $\rho = -1$) with a matched filter, Equation (3.68) can be written as

$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{3.70}$$



(a)



(b)

Figure 3.10 Binary signal vectors. (a) Antipodal. (b) Orthogonal.

Similarly, for the case of detecting orthogonal signals (that is, $\rho = 0$) with a matched filter, Equation (3.68) can be written as

$$P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{3.71}$$

Figure 3.10, where the signal magnitudes are each shown equal to $\sqrt{E_b}$, helps illustrate that the error performance described by Equations (3.70) and (3.71) is a function of the distance between $\mathbf{s}_1$ and $\mathbf{s}_2$ (the larger the distance, the smaller will be P_B). When the signals are antipodal, as in Figure 3.10a, the distance between them is $2\sqrt{E_b}$, and the energy E_d associated with that distance is characterized by the distance squared or $4E_b$. When we substitute $E_d = 4E_b$ into Equation (3.63), the result is Equation (3.70). When the signals are orthogonal as in Figure 3.10b, the distance between them is $\sqrt{2E_b}$ and thus $E_d = 2\ E_b$. When we substitute $E_d = 2\ E_b$ into Equation (3.63), the result is Equation (3.71).

Example 3.2 Matched Filter Detection of Antipodal Signals

Consider a binary communications system that receives equally likely signals $s_1(t)$ and $s_2(t)$ plus AWGN. See Figure 3.11. Assume that the receiving filter is a matched filter (MF), and that the noise-power spectral density N_0 is equal to 10^{-12} Watt/Hz. Use the values of received signal voltage and time shown on Figure 3.11 to compute the bit-error probability.

Solution

We can graphically determine the received energy per bit from the plot of either $s_1(t)$ or $s_2(t)$ shown in Figure 3.11 by integrating to find the energy (area under the voltage-squared pulse). Doing this in piecewise fashion, we get

$$\begin{aligned} E_b &= \int_0^3 v^2(t)\, dt \\ &= (10^{-3}\,\mathrm{V})^2 \times (10^{-6}\,\mathrm{s}) + (2 \times 10^{-3}\,\mathrm{V})^2 \times (10^{-6}\,\mathrm{s}) + (10^{-3}\,\mathrm{V})^2 \times (10^{-6}\,\mathrm{s}) \\ &= 6 \times 10^{-12}\,\text{joule} \end{aligned}$$

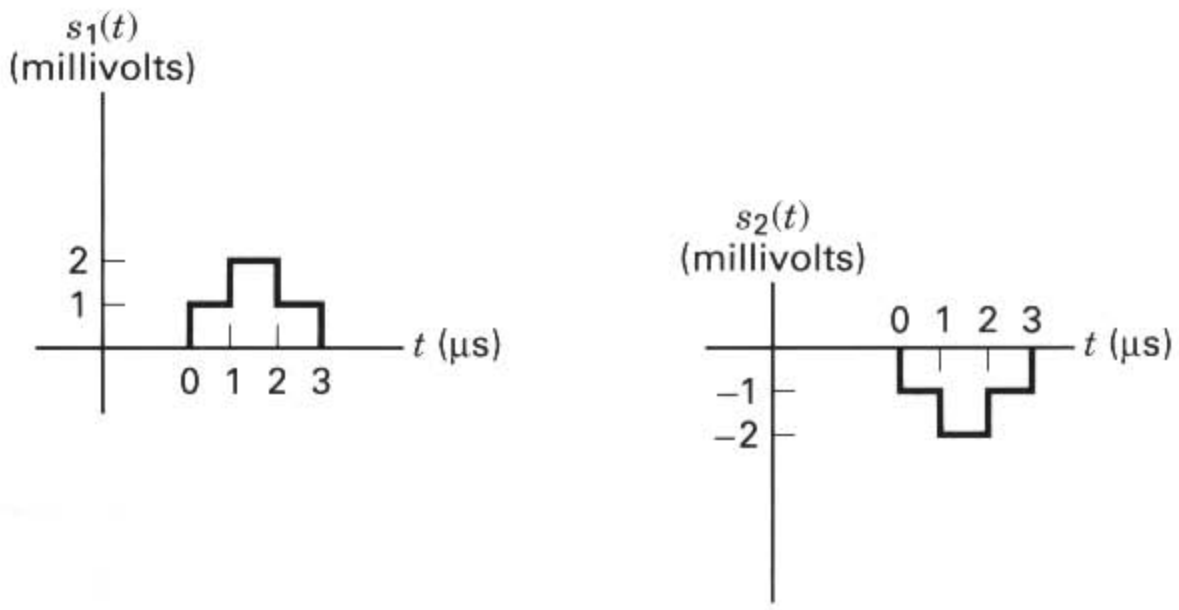


Figure 3.11 Baseband antipodal waveforms.

Since the waveforms depicted in Figure 3.11 are antipodal and are detected with a matched filter, we use Equation (3.70) to find the bit-error probability, as

$$Q\left(\sqrt{\frac{12 \times 10^{-12}}{10^{-12}}}\right) = Q(\sqrt{12}) = Q(3.46)$$

From Table B.1, we find that $P_B = 3 \times 10^{-4}$. Or, since the argument of $Q(x)$ is greater than 3, we can also use the approximate relationship in Equation (3.44) which yields $P_B \approx 2.9 \times 10^{-4}$.

Because the received signals are antipodal and are received by an MF, these are sufficient prerequisites such that Equation (3.70) provides the proper relationship for finding bit-error probability. The waveforms $s_1(t)$ and $s_2(t)$ could have been pictured in a much more bizarre fashion, but as long as they are antipodal and detected by an MF, their shapes do not enter into the P_B computations. The shapes of the waveforms, of course, *do matter* when it comes to specifying the impulse response of the MF needed to detect these waveforms.

3.2.5 Error Probability Performance of Binary Signaling

3.2.5.1 Unipolar Signaling

Figure 3.12a illustrates an example of baseband orthogonal signaling—namely, unipolar signaling, where

$$\begin{aligned} s_1(t) &= A \qquad 0 \le t \le T \qquad \text{for binary 1} \\ s_2(t) &= 0 \qquad 0 \le t \le T \qquad \text{for binary 0} \end{aligned} \tag{3.72}$$

and where $A > 0$ is the amplitude of symbol $s_1(t)$. The definition of orthogonal signaling described by Equation (3.69) requires that $s_1(t)$ and $s_2(t)$ have zero correlation over each symbol time duration. Because in Equation (3.72), $s_2(t)$ is equal to zero during the symbol time, this set of unipolar pulses clearly fulfills the condition

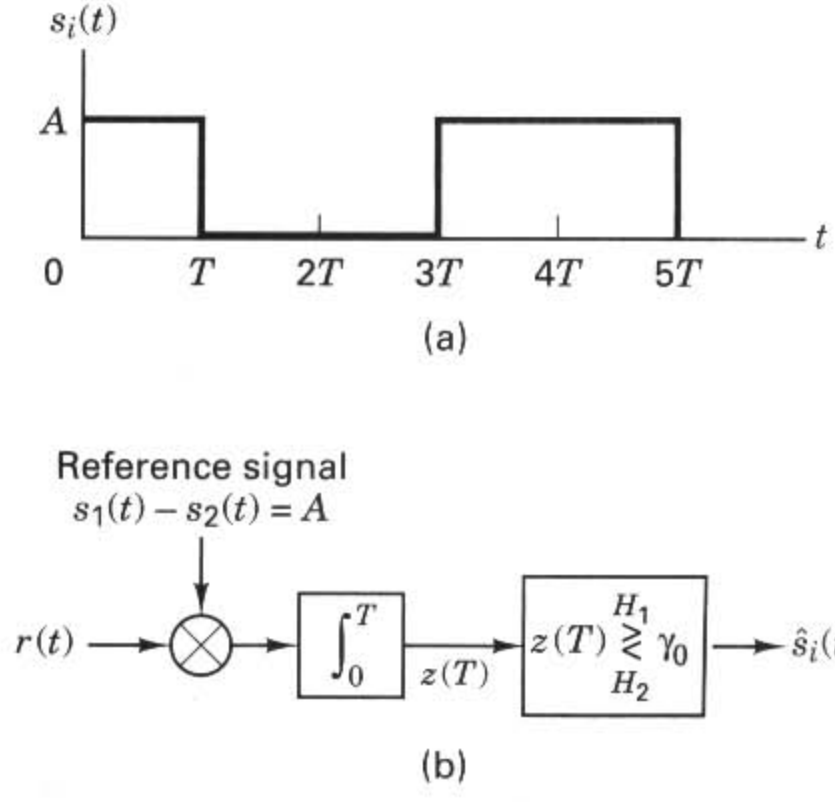


Figure 3.12 Detection of unipolar baseband signaling. (a) Unipolar signaling example. (b) Correlator detector.

shown in Equation (3.69), and hence, they form an orthogonal signal set. Consider such unipolar signaling, as illustrated in Figure 3.12a, as well as the correlator shown in Figure 3.12b, which can be used for detecting such pulses. The correlator multiplies and integrates the incoming signal $r(t)$ with the difference of the prototype signals, $[s_1(t) - s_2(t)] = A$. After a symbol duration T, a sampler (inferred by the upper limit of integration) yields the test statistic $z(T)$, which is then compared with the threshold γ_0. For the case of $s_1(t)$ plus AWGN being received—that is, when $r(t) = s_1(t) + n(t)$—the signal component of $z(T)$ is found, using Equation (3.59), to be

$$a_1(T) = \mathrm{E}\{z(T)|\mathrm{s}_1(t)\} = \mathrm{E}\left\{\int_0^T A^2 + An(t)\,dt\right\} = A^2T$$

where E $\{z(T)|\mathrm{s}_1(t)\}$ is the *expected value* of $z(T)$, given that $s_1(t)$ was sent. This follows since E $\{n(t)\} = 0$. Similarly, when $r(t) = s_2(t) + n(t)$, then $a_2(T) = 0$. Thus, in this case, the optimum decision threshold, from Equation (3.32), is given by $\gamma_0 = (a_1 + a_2)/2 = 1/2\ A^2T$. If the test statistic $z(T)$ is greater than γ_0, the signal is declared to be $s_1(t)$; otherwise, it is declared to be $s_2(t)$.

The energy difference signal, from Equation (3.62), is given by $E_d = A^2T$. Then the bit-error performance at the output is obtained from Equation (3.63) as

$$P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{3.73}$$

where, for the case of equally likely signaling, the *average* energy per bit is $E_b = A^2T/2$. Equation (3.73) corroborates Equation (3.71) where this relationship was established for orthogonal signaling in a more general way.

Note that the units out of a multiplier circuit, such as that seen in Figure 3.12b are volts. Therefore, for voltage signals on each of the two inputs, the multiplier transfer function must have units of 1/volt, and the measurable units of $r(t)\ s_i(t)$ out of the multiplier are volt/volt-squared. Similarly, the units out of an integrator circuit are also volts. Thus, for a voltage signal into an integrator, the integrator transfer function must have units of 1/second, and thus the overall transfer function of the product integrator has units of 1/volt-second. Then, for a signal into the product integrator having units of volt-squared-seconds (a measure of energy), we represent the output $z(T)$ as a voltage signal that is proportional to received signal energy (volt/joule).

3.2.5.2 Bipolar Signaling

Figure (3.13a) illustrates an example of baseband antipodal signaling—namely, bipolar signaling, where

$$s_1(t) = +A \qquad 0 \le t \le T \qquad \text{for binary 1}$$

and (3.74)

$$s_2(t) = -A \qquad 0 \le t \le T \qquad \text{for binary 0}$$

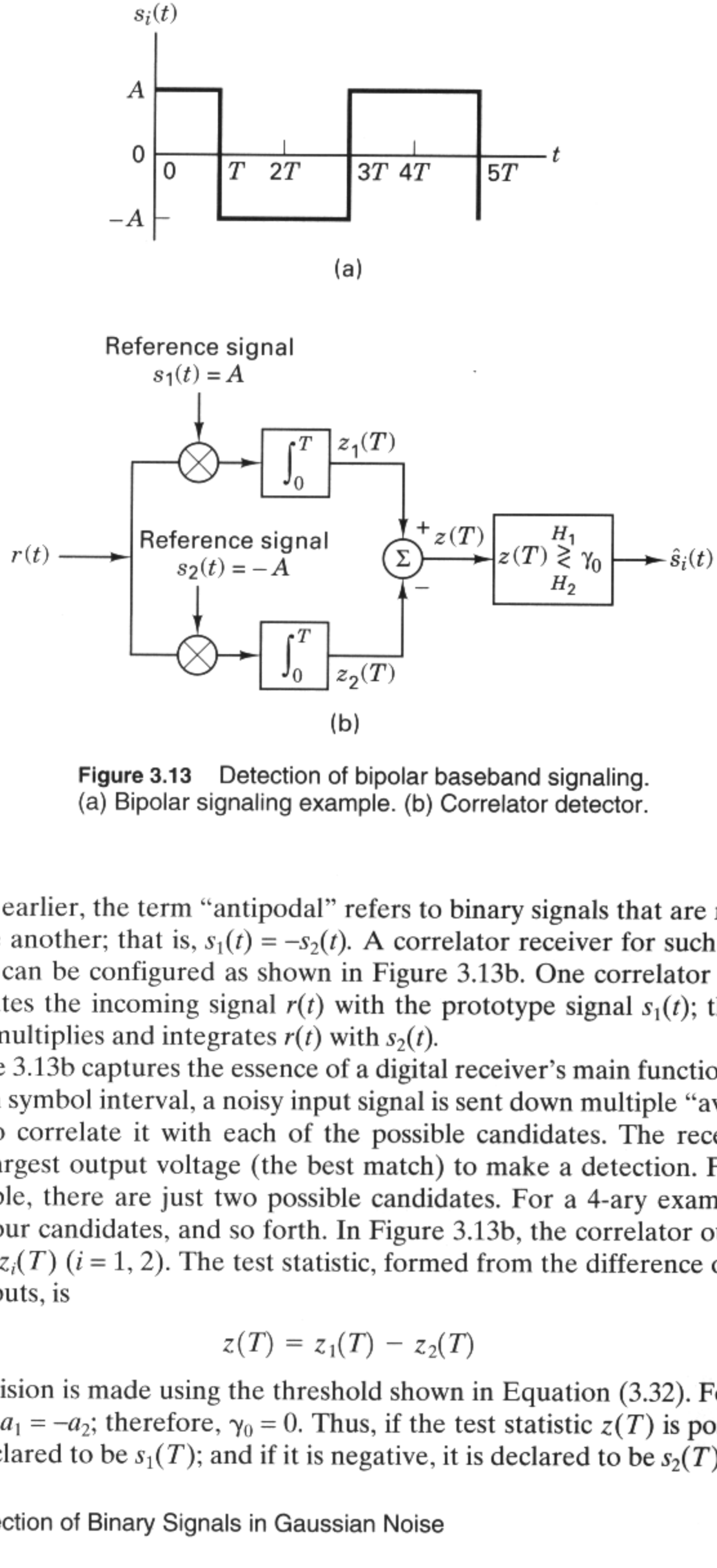


Figure 3.13 Detection of bipolar baseband signaling. (a) Bipolar signaling example. (b) Correlator detector.

As defined earlier, the term "antipodal" refers to binary signals that are mirror images of one another; that is, $s_1(t) = -s_2(t)$. A correlator receiver for such antipodal waveforms can be configured as shown in Figure 3.13b. One correlator multiplies and integrates the incoming signal $r(t)$ with the prototype signal $s_1(t)$; the second correlator multiplies and integrates $r(t)$ with $s_2(t)$.

Figure 3.13b captures the essence of a digital receiver's main function. That is, during each symbol interval, a noisy input signal is sent down multiple "avenues" in an effort to correlate it with each of the possible candidates. The receiver then seeks the largest output voltage (the best match) to make a detection. For the binary example, there are just two possible candidates. For a 4-ary example, there would be four candidates, and so forth. In Figure 3.13b, the correlator outputs are designated $z_i(T)$ $(i = 1, 2)$. The test statistic, formed from the difference of the correlator outputs, is

$$z(T) = z_1(T) - z_2(T) \tag{3.75}$$

and the decision is made using the threshold shown in Equation (3.32). For antipodal signals, $a_1 = -a_2$; therefore, $\gamma_0 = 0$. Thus, if the test statistic $z(T)$ is positive, the signal is declared to be $s_1(T)$; and if it is negative, it is declared to be $s_2(T)$.

From Equation (3.62), the energy-difference signal is $E_d = (2A)^2T$. Then, the bit-error performance at the output can be obtained from Equation (3.63) as

$$P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{3.76}$$

where the average energy per bit is given by $E_b = A^2T$. Equation (3.76) corroborates Equation (3.70) where this relationship was established for antipodal signaling in a more general way.

3.2.5.3 Signaling Described with Basis Functions

Instead of designating $s_i(t)$ as the reference signals in the correlator of Figure 3.13b, we can use the concept of *basis functions* described in Section 3.1.3. Binary signaling with unipolar or bipolar pulses provides particularly simple examples for doing this, because the entire signaling space can be described by just one basis function. If we normalize the space by choosing $K_j = 1$ in Equation (3.9), then it should be clear that the basis function $\psi_1(t)$ must be equal to $\sqrt{1/T}$.

For unipolar pulse signaling, we could then write

$$s_1(t) = a_{11}\psi_1(t) = A\sqrt{T} \times \left(\sqrt{\frac{1}{T}}\right) = A$$

and

$$s_2(t) = a_{21}\psi_1(t) = 0 \times \left(\sqrt{\frac{1}{T}}\right) = 0$$

where the coefficients a_{11} and a_{21} equal $A\sqrt{T}$ and 0, respectively.

For bipolar pulse signaling, we would write

$$s_1(t) = a_{11}\psi_1(t) = A\sqrt{T} \times \left(\sqrt{\frac{1}{T}}\right) = A$$

and

$$s_2(t) = a_{21}\psi_1(t) = -A\sqrt{T} \times \left(\sqrt{\frac{1}{T}}\right) = -A$$

where the coefficients a_{11} and a_{21} equal $A\sqrt{T}$ and $-A\sqrt{T}$, respectively. For the case of antipodal pulses, we can envision the correlator receiver taking the form of Figure 3.12b with the reference signal equal to $\sqrt{1/T}$. Then, for the case of $s_1(t) = A$ being sent, we can write

$$a_1(T) = \mathbf{E}\{z(T)|s_1(t)\} = \mathrm{E}\left\{\int_0^T \frac{A}{\sqrt{T}} + \frac{n(t)}{\sqrt{T}}\,dt\right\} = A\sqrt{T}$$

This follows because $\mathbf{E}\{n(t)\} = 0$, and since, for antipodal signaling, $E_b = A^2T$, it follows that $a_1(T) = \sqrt{E_b}$. Similarly, for a received signal $r(t) = s_2(t) + n(t)$, it follows that $a_2(T) = -\sqrt{E_b}$. When reference signals are treated in this way, then the ex-

pected value of $z(T)$ has a magnitude of $\sqrt{E_b}$, which has units of normalized volts proportional to received energy. This basis-function treatment of the correlator yields a convenient value of $z(T)$ that is consistent with units of volts out of multipliers and integrators. We therefore repeat an important point: At the output of the sampler (the predetection point), the test statistic $z(T)$ is a voltage signal that is proportional to received signal energy.

Figure 3.14 illustrates curves of P_B versus E_b/N_0 for bipolar and unipolar signaling. There are only two fair ways to compare such curves. By drawing a vertical line at some given E_b/N_0, say, 10 dB, we see that the unipolar signaling yields P_B in the order of 10^{-3}, but that the bipolar signaling yields P_B in the order of 10^{-6}. The lower curve is the better performing one. Also, by drawing a horizontal line at some required P_B, say 10^{-5}, we see that with unipolar signaling each received bit would require an E_b/N_0 of about 12.5 dB, but with bipolar signaling, we could get away with requiring each received bit to have an E_b/N_0 of only about 9.5 dB. Of course, the smaller requirement of E_b/N_0 is better (using less power, smaller batteries). In general, the better performing curves are the ones closest to the axes, lower and leftmost. In examining the two curves in Figure 3.14, we can see a 3-dB error-performance improvement for bipolar compared with unipolar signaling. This

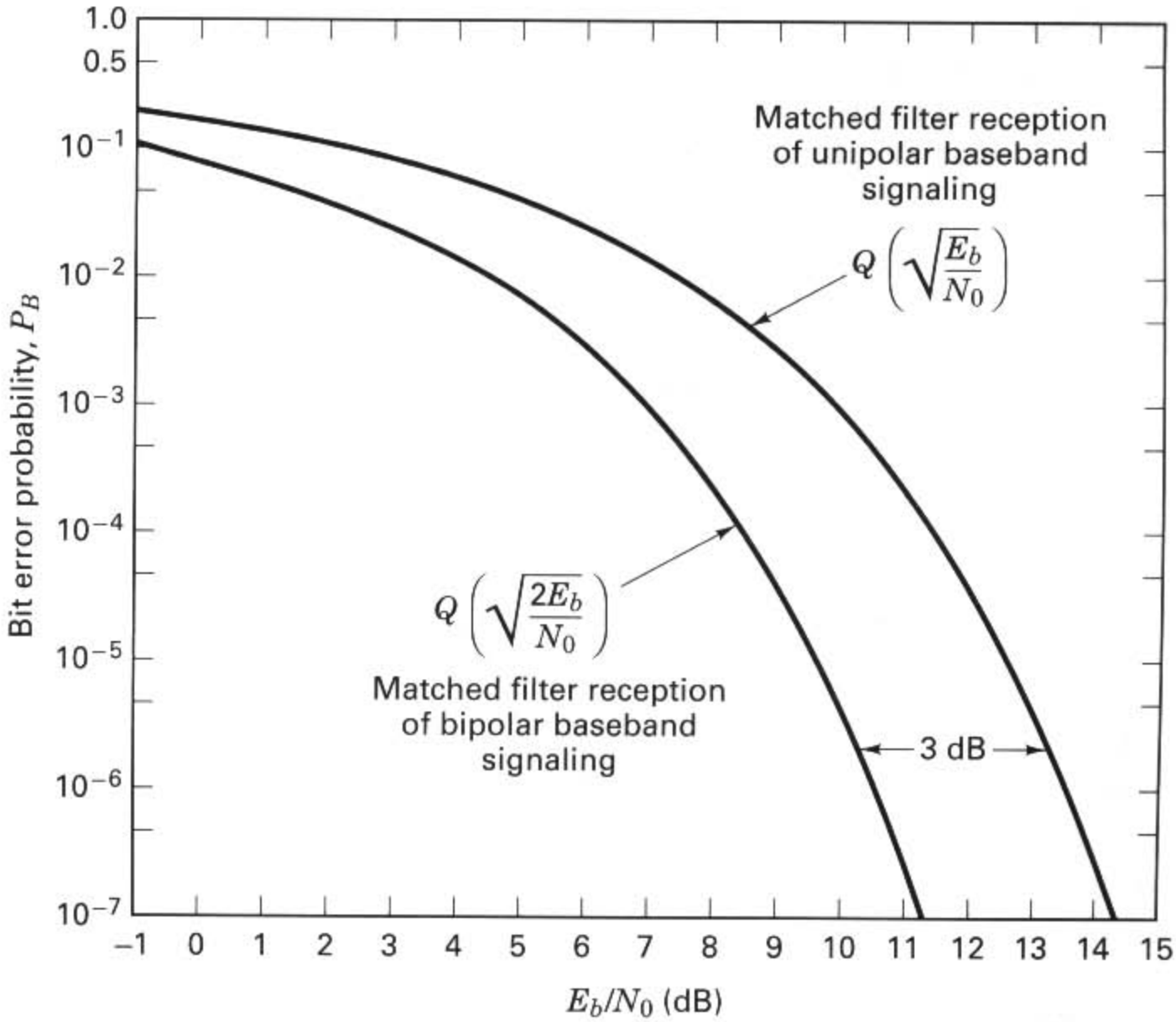


Figure 3.14 Bit error performance of unipolar and bipolar signaling.

difference could have been predicted by the factor-of-2 difference in the coefficient of E_b/N_0 in Equation (3.70) compared with (3.71). In Chapter 4, it is shown that, with MF detection, *bandpass* antipodal signaling (e.g., binary phase-shift keying) has the same P_B performance as *baseband* antipodal signaling (e.g., bipolar pulses). It is also shown that, with MF detection, *bandpass* orthogonal signaling (e.g., orthogonal frequency-shift keying) has the same P_B performance as *baseband* orthogonal signaling (e.g., unipolar pulses).

3.3 INTERSYMBOL INTERFERENCE

Figure 3.15a introduces the filtering aspects of a typical digital communication system. There are various filters (and reactive circuit elements such as inductors and capacitors) throughout the system—in the transmitter, in the receiver, and in the channel. At the transmitter, the information symbols, characterized as impulses or voltage levels, modulate pulses that are then filtered to comply with some bandwidth constraint. For baseband systems, the channel (a cable) has distributed reactances that distort the pulses. Some bandpass systems, such as wireless systems, are characterized by fading channels (see Chapter 15), that behave like undesirable filters manifesting signal distortion. When the receiving filter is configured to compensate for the distortion caused by *both* the transmitter and the channel, it is often referred to as an *equalizing filter* or a *receiving/equalizing filter.* Figure 3.15b illustrates a convenient model for the system, lumping all the filtering effects into one overall equivalent system transfer function

$$H(f) = H_t(f)\, H_c(f)\, H_r(f) \tag{3.77}$$

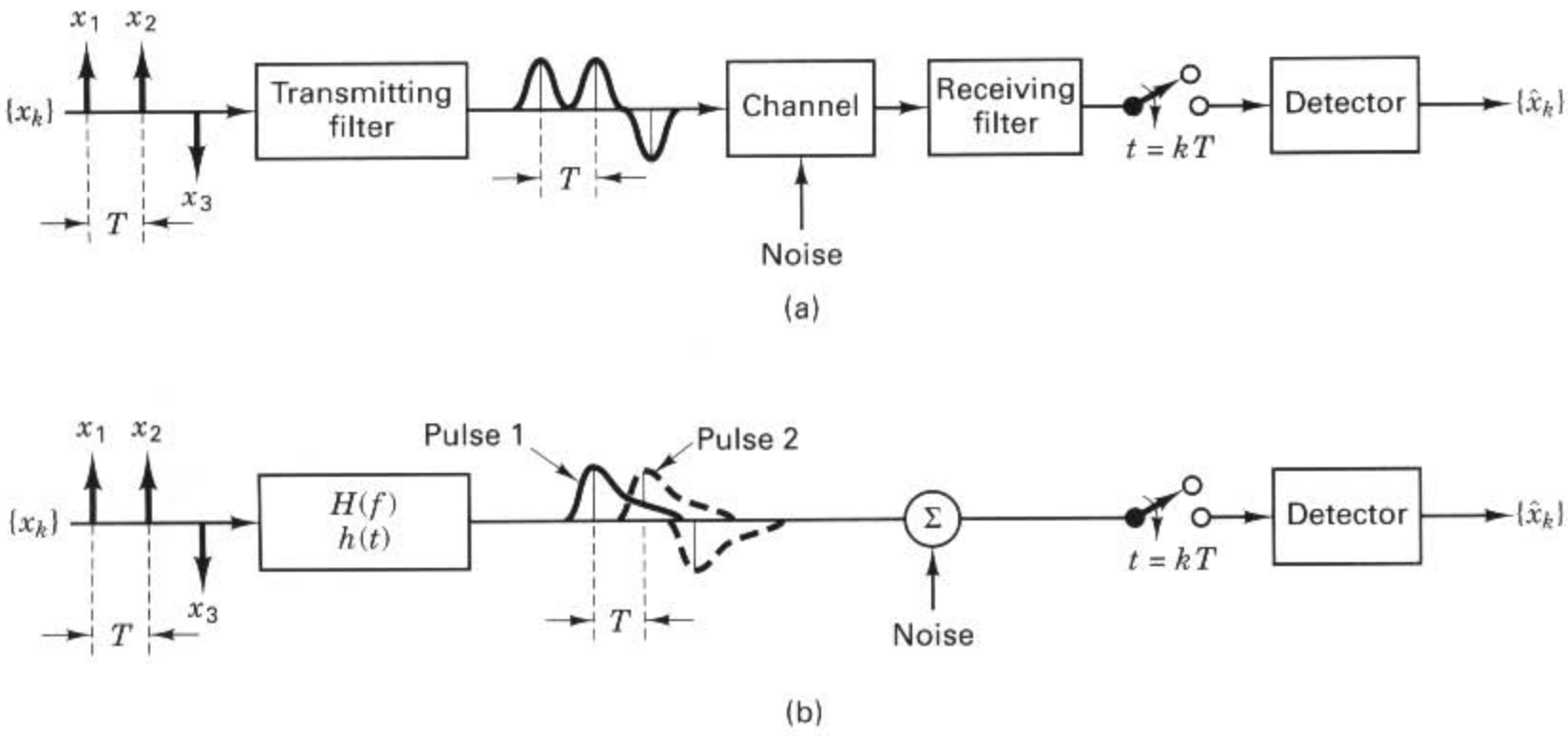


Figure 3.15 Intersymbol interference in the detection process. (a) Typical baseband digital system. (b) Equivalent model.

where $H_t(f)$ characterizes the transmitting filter, $H_c(f)$ the filtering within the channel, and $H_r(f)$ the receiving/equalizing filter. The characteristic $H(f)$, then, represents the composite system transfer function due to all the filtering at various locations throughout the transmitter/channel/receiver chain. In a binary system with a common PCM waveform, such as NRZ-L, the detector makes a symbol decision by comparing a sample of the received pulse to a threshold; for example, the detector in Figure 3.15 decides that a binary one was sent if the received pulse is positive, and that a binary zero was sent, if the received pulse is negative. Due to the effects of system filtering, the received pulses can overlap one another as shown in Figure 3.15b. The tail of a pulse can "smear" into adjacent symbol intervals, thereby interfering with the detection process and degrading the error performance; such interference is termed *intersymbol interference* (ISI). Even in the absence of noise, the effects of filtering and channel-induced distortion lead to ISI. Sometimes $H_c(f)$ is specified, and the problem remains to determine $H_t(f)$ and $H_r(f)$, such that the ISI is minimized at the output of $H_r(f)$.

Nyquist [6] investigated the problem of specifying a received pulse shape so that no ISI occurs at the detector. He showed that the theoretical minimum system bandwidth needed in order to detect R_s symbols/s, without ISI, is $R_s/2$ hertz. This occurs when the system transfer function $H(f)$ is made rectangular, as shown in Figure 3.16a. For baseband systems, when $H(f)$ is such a filter with single-sided bandwidth $1/2T$ (the *ideal Nyquist filter*), its impulse response, the inverse Fourier transform of $H(f)$ (from Table A.1) is of the form $h(t) = \text{sinc}\,(t/T)$, shown in Figure 3.16b. This sinc (t/T)-shaped pulse is called the *ideal Nyquist pulse;* its multiple lobes comprise a mainlobe and sidelobes called pre- and post-mainlobe *tails* that are infinitely long. Nyquist established that if each pulse of a received sequence is of the form sinc (t/T), the pulses can be detected without ISI. Figure 3.16b illustrates how ISI is avoided. There are two successive pulses, $h(t)$ and $h(t - T)$. Even though $h(t)$ has long tails, the figure shows a tail passing through zero amplitude at the instant $(t = T)$ when $h\ (t - T)$ is to be sampled, and likewise all tails pass through zero amplitude when any other pulse of the sequence $h(t - kT)$, $k = \pm 1$, ± 2, . . . is to be sampled. Therefore, assuming that the sample timing is perfect, there will be no ISI degradation introduced. For baseband systems, the bandwidth

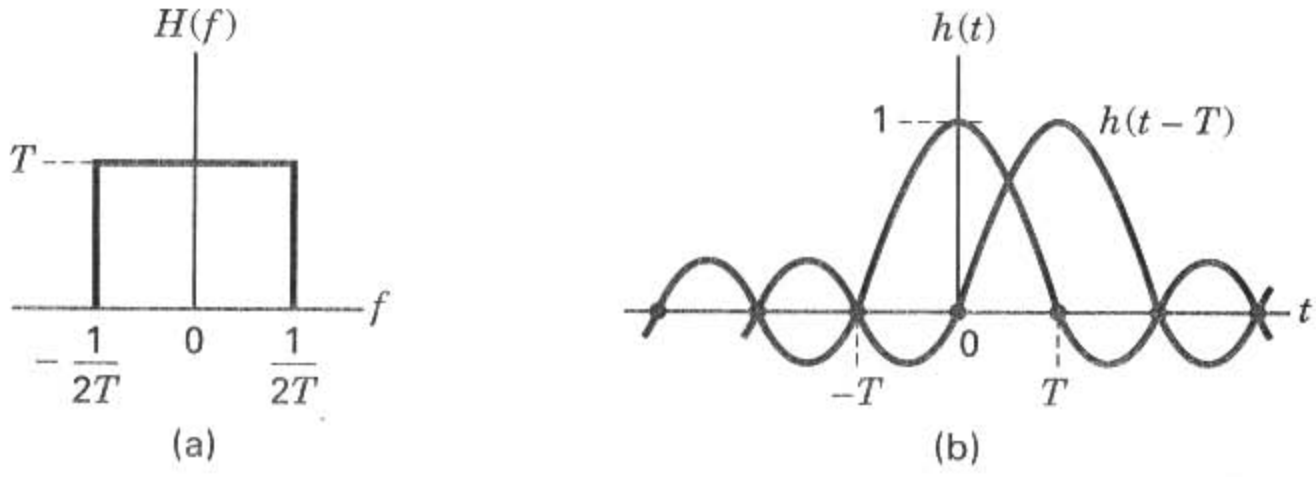


Figure 3.16 Nyquist channels for zero ISI. (a) Rectangular system transfer function $H(f)$. (b) Received pulse shape $h(t) = \text{sinc}\,(t/T)$.

required to detect $1/T$ such pulses (symbols) per second is equal to $1/2T$; in other words, a system with bandwidth $W = 1/2T = R_s/2$ hertz can support a maximum transmission rate of $2W = 1/T = R_s$ symbols/s (*Nyquist bandwidth constraint*) without ISI. Thus, for ideal Nyquist filtering (and zero ISI), the maximum possible symbol transmission rate per hertz, called the *symbol-rate packing,* is 2 symbols/s/Hz. It should be clear from the rectangular-shaped transfer function of the ideal Nyquist filter and the infinite length of its corresponding pulse, that such ideal filters are not realizable; they can only be approximately realized.

The names "Nyquist filter" and "Nyquist pulse" are often used to describe the general class of filtering and pulse-shaping that satisfy zero ISI at the sampling points. A Nyquist filter is one whose frequency transfer function can be represented by a rectangular function convolved with any real even-symmetric frequency function. A Nyquist pulse is one whose shape can be represented by a sinc (t/T) function multiplied by another time function. Hence, there are a countless number of Nyquist filters and corresponding pulse shapes. Amongst the class of Nyquist filters, the most popular ones are the raised cosine and the root-raised cosine, treated below.

A fundamental parameter for communication systems is *bandwidth efficiency, R/W,* whose units are bits/s/Hz. As the units imply, R/W represents a measure of data throughput per hertz of bandwidth and thus measures how efficiently any signaling technique utilizes the bandwidth resource. Since the Nyquist bandwidth constraint dictates that the theoretical maximum symbol-rate packing without ISI is 2 symbols/s/Hz, one might ask what it says about the maximum number of bits/s/Hz. It says nothing about bits, directly; the constraint deals only with pulses or symbols, and the ability to detect their amplitude values without distortion from other pulses. To find R/W for any signaling scheme, one must know how many bits each symbol represents, which is a separate issue. Consider an M-ary PAM signaling set. Each symbol (comprising k bits) is represented by one of M-pusle amplitudes. For $k = 6$ bits per symbol, the symbol set size is $M = 2^k = 64$ amplitudes. Thus with 64-ary PAM, the theoretical maximum bandwidth efficiency that is possible without ISI is 12 bits/s/Hz. (Bandwidth efficiency is treated in greater detail in Chapter 9.)

3.3.1 Pulse Shaping to Reduce ISI

3.3.1.1 Goals and Trade-offs

The more compact we make the signaling spectrum, the higher is the allowable data rate or the greater is the number of users that can simultaneously be served. This has important implications to communication service providers, since greater utilization of the available bandwidth translates into greater revenue. For most communication systems (with the exception of spread-spectrum systems, covered in Chapter 12), our goal is to reduce the required system bandwidth as much as possible. Nyquist has provided us with a basic limitation to such bandwidth reduction. What would happen if we tried to force a system to operate at smaller bandwidths than the constraint dictates? The pulses would become spread in time,

which would degrade the system's error performance due to increased ISI. A prudent goal is to compress the bandwidth of the data impulses to some reasonably small bandwidth greater than the Nyquist minimum. This is accomplished by pulse-shaping with a Nyquist filter. If the band edge of the filter is steep, approaching the rectangle in Figure 3.16a, then the signaling spectrum can be made most compact. However, such a filter has an impulse response duration approaching infinity, as indicated in Figure 3.16b. Each pulse extends into every pulse in the entire sequence. Long time responses exhibit large-amplitude tails nearest the main lobe of each pulse. Such tails are undesirable because, as shown in Figure 3.16b, they contribute zero ISI *only* when the sampling is performed *at exactly* the correct sampling time; when the tails are large, small timing errors will result in ISI. Therefore, although a compact spectrum provides optimum bandwidth utilization, it is very susceptible to ISI degradation induced by timing errors.

3.3.1.2 The Raised-Cosine Filter

Earlier, it was stated that the receiving filter is often referred to as an *equalizing filter,* when it is configured to compensate for the distortion caused by both the transmitter and the channel. In other words, the configuration of this filter is chosen so as to optimize the composite system frequency transfer function $H(f)$, shown in Equation (3.77). One frequently used $H(f)$ transfer function belonging to the Nyquist class (zero ISI at the sampling times) is called the *raised-cosine filter.* It can be expressed as

$$H(f) = \begin{cases} 1 & \text{for } |f| < 2W_0 - W \\ \cos^2\left(\dfrac{\pi}{4}\dfrac{|f| + W - 2W_0}{W - W_0}\right) & \text{for } 2W_0 - W < |f| < W \\ 0 & \text{for } |f| > W \end{cases} \quad (3.78)$$

where W is the absolute bandwidth and $W_0 = 1/2T$ represents the minimum Nyquist bandwidth for the rectangular spectrum and the –6-dB bandwidth (or half-amplitude point) for the raised-cosine spectrum. The difference $W - W_0$ is termed the "excess bandwidth," which means additional bandwidth beyond the Nyquist minimum (i.e., for the rectangular spectrum, W is equal to W_0). The *roll-off factor* is defined to be $r = (W - W_0)/W_0$, where $0 \le r \le 1$. It represents the excess bandwidth divided by the filter –6-dB bandwidth (i.e., the fractional excess bandwidth). For a given W_0, the roll-off r specifies the required excess bandwidth as a fraction of W_0 and characterizes the steepness of the filter roll off. The raised-cosine characteristic is illustrated in Figure 3.17a for roll-off values of $r = 0$, $r = 0.5$, and $r = 1$. The $r = 0$ roll-off is the Nyquist minimum-bandwidth case. Note that when $r = 1$, the required excess bandwidth is 100%, and the tails are quite small. A system with such an overall spectral characteristic can provide a symbol rate of R_s symbols/s using a bandwidth of R_s hertz (twice the Nyquist minimum bandwidth), thus yielding a symbol-rate packing of 1 symbol/s/Hz. The corresponding impulse response for the $H(f)$ of Equation (3.78) is

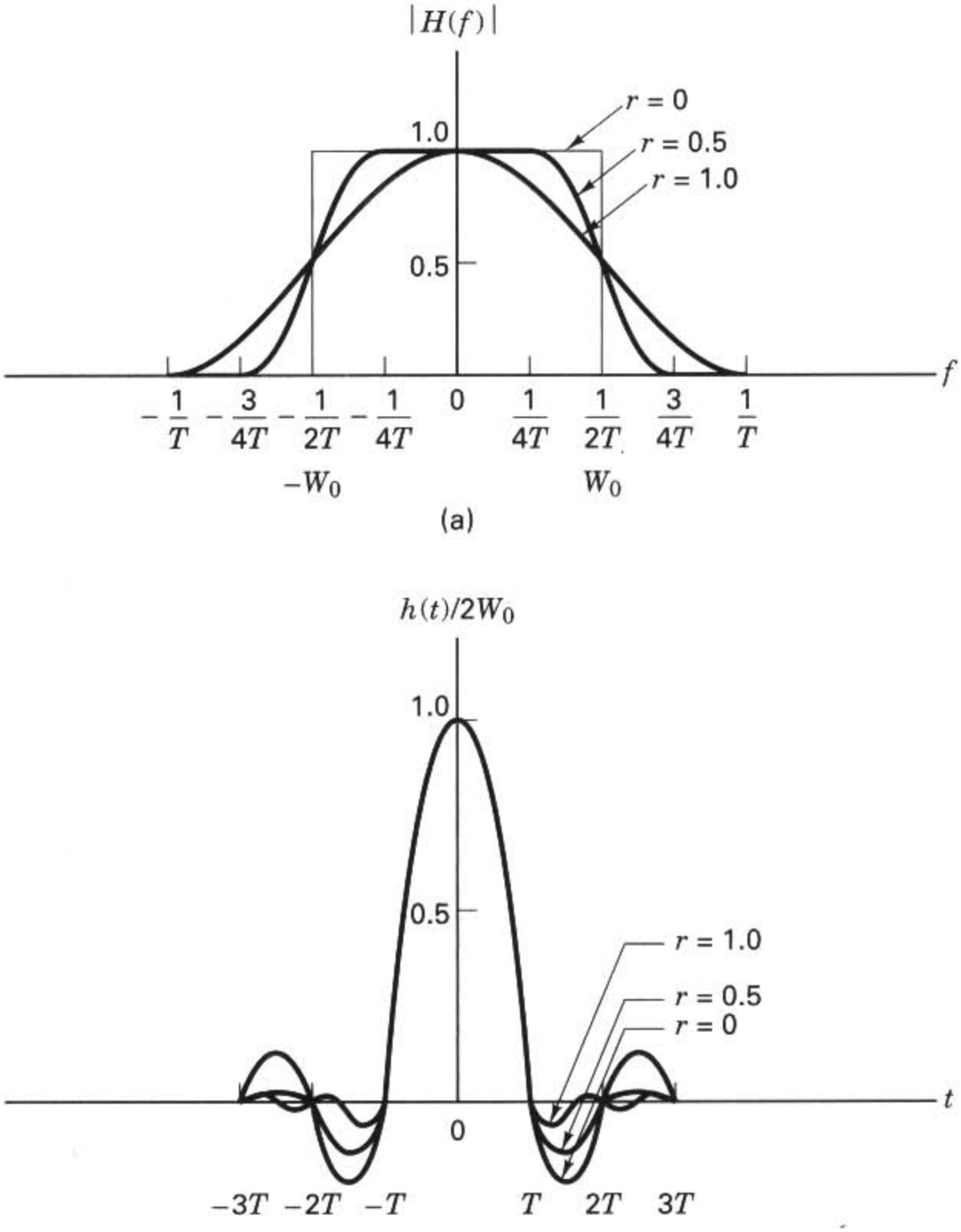


Figure 3.17 Raised-cosine filter characteristics. (a) System transfer function. (b) System impulse response.

$$h(t) = 2W_0(\text{sinc } 2W_0t)\frac{\cos\left[2\pi(W - W_0)t\right]}{1 - \left[4(W - W_0)t\right]^2} \tag{3.79}$$

and is plotted in Figure 3.17b for $r = 0$, $r = 0.5$, and $r = 1$. The tails have zero value at each pulse-sampling time, regardless of the roll-off value.

We can only approximately implement a filter described by Equation (3.78) and a pulse shape described by Equation (3.79), since, strictly speaking, the raised-cosine spectrum is not physically realizable (for the same reason that the ideal Nyquist filter is not realizable). A realizable filter must have an impulse response of finite duration and exhibit a zero output prior to the pulse turn-on time (see Sec-

tion 1.7.2), which is not the case for the family of raised-cosine characteristics. These unrealizable filters are *noncausal* (the filter impulse response has infinite duration, and the filtered pulse begins at time $t = -\infty$). A pulse-shaping filter should satisfy two requirements. It should provide the desired roll-off, and it should be realizable (the impulse response needs to be truncated to a finite length).

Starting with the Nyquist bandwidth constraint that the minimum required system bandwidth W for a symbol rate of R_s symbols/s without ISI is $R_s/2$ hertz, a more general relationship between required bandwidth and symbol transmission rate involves the filter roll-off factor r and can be stated as

$$W = \frac{1}{2}(1 + r)R_s \tag{3.80}$$

Thus, with $r = 0$, Equation (3.80) describes the minimum required bandwidth for ideal Nyquist filtering. For $r > 0$, there is a bandwidth expansion beyond the Nyquist minimum; thus, for this case, R_s is now less than twice the bandwidth. If the demodulator outputs one sample per symbol, then the Nyquist sampling theorem has been violated, since we are left with too few samples to reconstruct the analog waveform unambiguously (aliasing is present). However, for digital communication systems, we are not interested in reconstructing the analog waveform. Since the family of raised-cosine filters is characterized by zero ISI at the times that the symbols are sampled, we can still achieve unambiguous detection.

Bandpass-modulated signals (see Chapter 4), such as amplitude shift keying (ASK) and phase-shift keying (PSK), require twice the transmission bandwidth of the equivalent baseband signals. (See Section 1.7.1.) Such frequency-translated signals, occupying twice their baseband bandwidth, are often called double-sideband (DSB) signals. Therefore, for ASK- and PSK-modulated signals, the relationship between the required DSB bandwidth W_{DSB} and the symbol transmission rate R_s is

$$W_{\text{DSB}} = (1 + r)R_s \tag{3.81}$$

Recall that the raised-cosine frequency transfer function describes the composite $H(f)$ that is the "full round trip" from the inception of the message (as an impulse) at the transmitter, through the channel, and through the receiving filter. The filtering at the receiver is the compensating portion of the overall transfer function to help bring about zero ISI with an overall transfer function, such as the raised cosine. Often this is accomplished by choosing (matching) the receiving filter and the transmitting filter so that each has a transfer function known as a root-raised cosine (square root of the raised cosine). Neglecting any channel-induced ISI, the product of these root-raised-cosine functions yields the composite raised-cosine system transfer function. Whenever a separate equalizing filter is introduced to mitigate the effects of channel-induced ISI, the receiving and equalizing filters together should be configured to compensate for the distortion caused by both the transmitter and the channel so as to yield an overall system transfer function characterized by zero ISI.

Let's review the trade-off that faces us in specifying pulse-shaping filters. The larger the filter roll-off, the shorter will be the pulse tails (which implies smaller tail

amplitudes). Small tails exhibit less sensitivity to timing errors and thus make for small degradation due to ISI. Notice in Figure 3.17b, for $r = 1$, that timing errors can still result in some ISI degradation. However, the problem is not as serious as it is for the case in which $r = 0$, because the tails of the $h(t)$ waveform are of much smaller amplitude for $r = 1$ than they are for $r = 0$. The cost is more excess bandwidth. On the other hand, the smaller the filter roll-off, the smaller will be the excess bandwidth, thereby allowing us to increase the signaling rate or the number of users that can simultaneously use the system. The cost is longer pulse tails, larger pulse amplitudes, and thus, greater sensitivity to timing errors.

3.3.2 Two Types of Error-Performance Degradation

The effects of error-performance degradation in digital communications can be partitioned into two categories. The first is due to a decrease in received signal power or an increase in noise or interference power, giving rise to a loss in signal-to-noise ratio or E_b/N_0. The second is due to signal distortion, such as might be caused by intersymbol interference (ISI). Let us demonstrate how different are the effects of these two degradation types.

Suppose that we require a communication system with a bit-error probability P_B versus E_b/N_0 characteristic corresponding to the solid-line curve plotted in Figure 3.18a. Suppose that after the system is configured and measurements are taken, we find, to our disappointment, that the performance does not follow the theoretical curve, but in fact follows the dashed line plot. A loss in E_b/N_0 has come about because of some signal losses or an increased level of noise or interference. For a

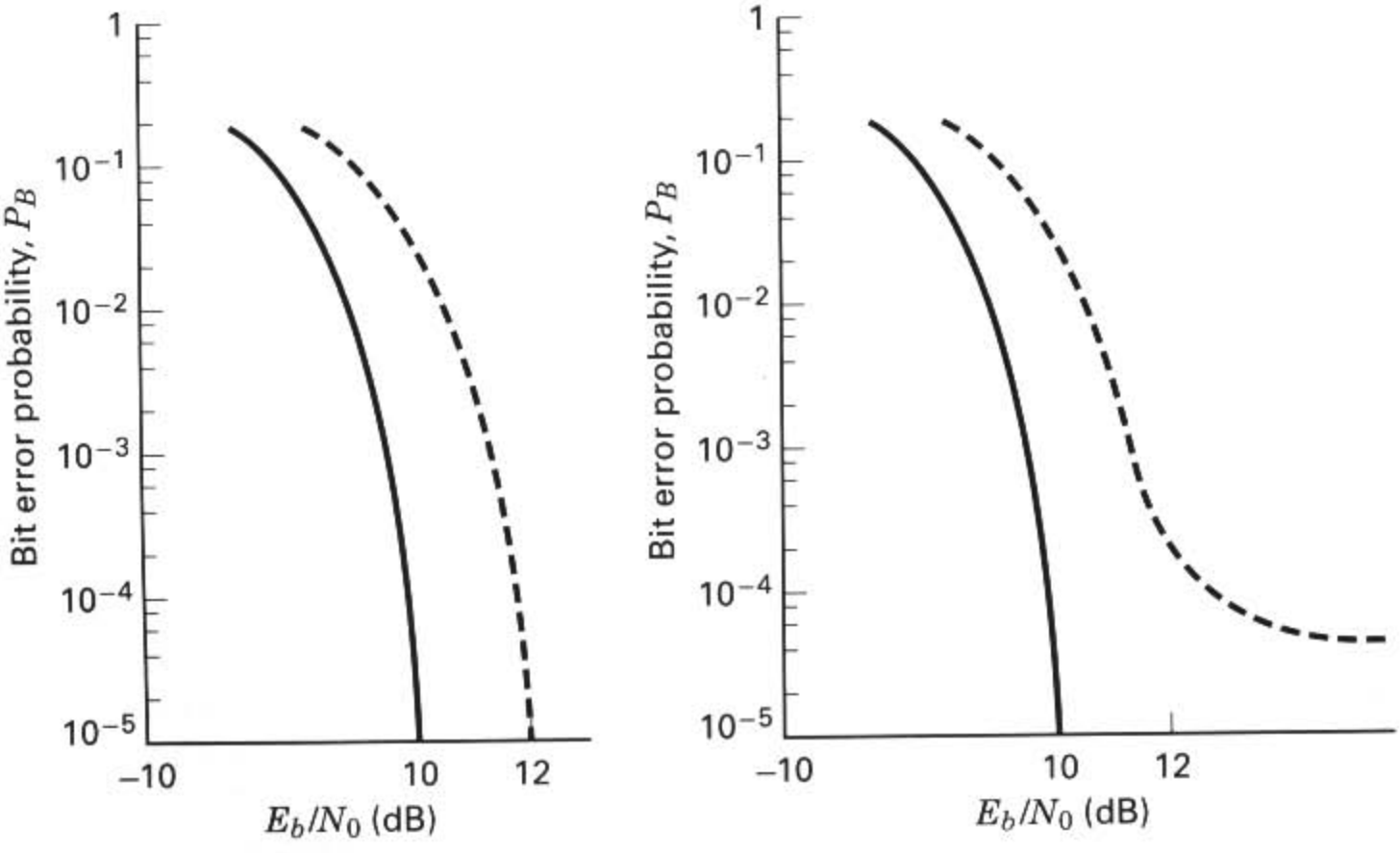


Figure 3.18 (a) Loss in E_b/N_0. (b) Irreducible P_B caused by distortion.

desired bit-error probability of 10^{-5}, the theoretical required E_b/N_0 is 10 dB. Since our system performance falls short of our goal, we can see from the dashed-line curve that, for the same bit-error probability of 10^{-5}, the required E_b/N_0 is now 12 dB. If there were no way to remedy this problem, how much more E_b/N_0 would have to be provided in order to meet the required bit-error probability? The answer is 2 dB, of course. It might be a serious problem—especially if the system is power-limited, and it is difficult to come up with the additional 2 dB. But that loss in E_b/N_0 is not *so* terrible when compared with the possible effects of degradation caused by a distortion mechanism.

In Figure 3.18b, again imagine that we do not meet the desired performance of the solid-line curve. But instead of suffering a simple loss in signal-to-noise ratio, there is a degradation effect brought about by ISI (plotted with the dashed line). If there were no way to remedy this problem, how much more E_b/N_0 would be required in order to meet the desired bit-error probability? It would require an infinite amount—or, in other words, there is no amount of E_b/N_0 that will ameliorate this problem. More E_b/N_0 cannot help when the curve manifests such an irreducible P_B (assuming that the bottoming-out point is located above the system's required P_B). Undoubtedly, every P_B-versus-E_b/N_0 curve bottoms out somewhere, but if the bottoming-out point is well below the region of interest, it will be of no consequence.

More E_b/N_0 may not help the ISI problem (it won't help at all if the P_B curve has reached an irreducible level). This can be inferred by looking at the overlapped pulses in Figure 3.15b; if we increase the E_b/N_0, the ratio of that overlap does not change. The pulses are subject to the same distortion. What, then, is the usual cure for the degradation effects of ISI? The cure is found in a technique called equalization. (See Section 3.4.) Since the distortion effects of ISI are caused by filtering in the transmitter and the channel, equalization can be thought of as the process that reverses such nonoptimum filtering effects.

Example 3.3 Bandwidth Requirements

(a) Find the minimum required bandwidth for the baseband transmission of a four-level PAM pulse sequence having a data rate of $R = 2400$ bits/s if the system transfer characteristic consists of a raised-cosine spectrum with 100% excess bandwidth ($r = 1$).

(b) The same 4-ary PAM sequence is modulated onto a carrier wave, so that the baseband spectrum is shifted and centered at frequency f_0. Find the minimum required DSB bandwidth for transmitting the modulated PAM sequence. Assume that the system transfer characteristic is the same as in part (a).

Solution

(a) $M = 2^k$; since $M = 4$ levels, $k = 2$.

$$\text{Symbol or pulse rate } R_s = \frac{R}{k} = \frac{2400}{2} = 1200 \text{ symbols/s;}$$

$$\text{Minimum bandwidth } W = \tfrac{1}{2}(1 + r)R_s = \tfrac{1}{2}(2)(1200) = 1200 \text{ Hz.}$$

Figure 3.19a illustrates the baseband PAM received pulse in the time domain—an approximation to the $h(t)$ in Equation (3.79). Figure 3.19b illustrates the Fourier

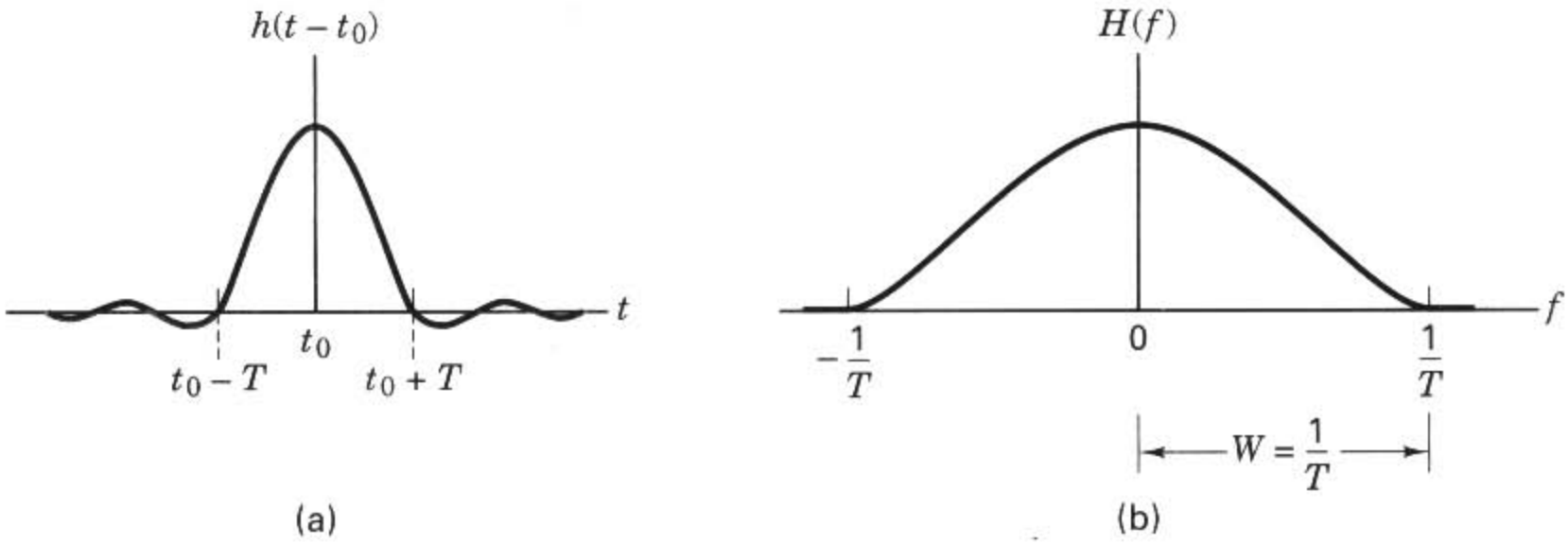


Figure 3.19 (a) Shaped pulse. (b) Baseband raised cosine spectrum.

transform of $h(t)$—the raised cosine spectrum. Notice that the required bandwidth, W, starts at zero frequency and extends to $f = 1/T$; it is twice the size of the Nyquist theoretical minimum bandwidth.

(b) As in part (a),

$$R_s = 1200 \text{ symbols/s};$$

$$W_{\text{DSB}} = (1 + r)R_s = 2(1200) = 2400 \text{ Hz}.$$

Figure 3.20a illustrates the modulated PAM received pulse. This waveform can be viewed as the product of a high-frequency sinusoidal carrier wave and a waveform with the pulse shape of Figure 3.19a. The single-sided spectral plot in Figure 3.20b illustrates that the modulated bandwidth is

$$W_{\text{DSB}} = \left(f_0 + \frac{1}{T}\right) - \left(f_0 - \frac{1}{T}\right) = \frac{2}{T}.$$

When the spectrum of Figure 3.19b is shifted up in frequency, the negative and positive halves of the baseband spectrum are shifted up in frequency, thereby dou-

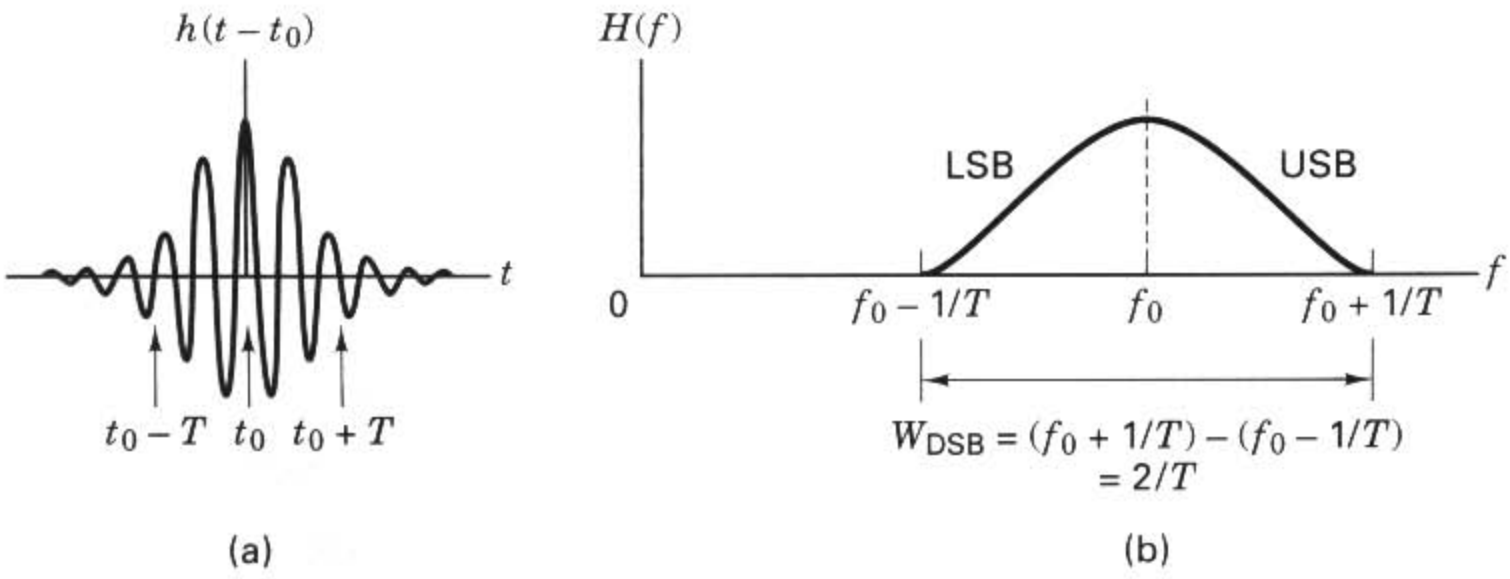


Figure 3.20 (a) Modulated shaped pulse. (b) DSB-modulated raised cosine spectrum.

bling the required transmission bandwidth. As the name implies, the DSB signal has two sidebands: the upper sideband (USB), derived from the baseband positive half, and the lower sideband (LSB), derived from the baseband negative half.

Example 3.4 Digital Telephone Circuits

Compare the system bandwidth requirements for a terrestrial 3-kHz analog telephone voice channel with that of a digital one. For the digital channel, the voice is formatted as a PCM bit stream, where the sampling rate for the analog-to-digital (A/D) conversion is 8000 samples/s and each voice sample is quantized to one of 256 levels. The bit stream is then transmitted using a PCM waveform and received with zero ISI.

Solution

The result of the sampling and quanitization process yields PCM words such that each word (representing one sample) has one of $L = 256$ different levels. If each sample were sent as a 256-ary PAM pulse (symbol), then from Equation (3.80) we can write that the required system bandwidth (without ISI) for sending R_s symbols/s would be

$$W \geq \frac{R_s}{2} \text{ hertz}$$

where the equality sign holds true only for ideal Nyquist filtering. Since the digital telephone system uses PCM (binary) waveforms, each PCM word is converted to $\ell = \log_2 L = \log_2 256 = 8$ bits. Therefore, the system bandwidth required to transmit voice using PCM is

$$W_{\text{PCM}} \geq (\log_2 L)\frac{R_s}{2} \text{ hertz}$$

$$\geq \frac{1}{2}(8 \text{ bits/symbol})\,(8000 \text{ symbols/s}) = 32 \text{ kHz}$$

The 3-kHz analog voice channel will generally require approximately 4-kHz of bandwidth, including some bandwidth separation between channels, called *guard bands.* Therefore, the PCM format, using 8-bit quantization and binary signaling with a PCM waveform, requires at least eight times the bandwidth required for the analog channel.

3.3.3 Demodulation/Detection of Shaped Pulses

3.3.3.1 Matched Filters versus Conventional Filters

Conventional filters screen out unwanted spectral components of a received signal while maintaining some measure of fidelity for signals occupying a selected span of the spectrum, called the *pass-band.* These filters are generally designed to provide approximately uniform gain, a linear phase-versus-frequency characteristic over the pass-band, and a specified minimum attenuation over the remaining spectrum, called the *stop-band(s).* A matched filter has a different "design priority," namely that of maximizing the SNR of a known signal in the presence of AWGN. Conventional filters are applied to random signals defined only by their bandwidth, while matched filters are applied to *known signals* with random parameters (such as amplitude and arrival time). The matched filter can be considered to be a *template* that is matched to the known shape of the signal being processed. A conven-

tional filter tries to preserve the temporal or spectral structure of the signal of interest. On the other hand, a matched filter significantly modifies the temporal structure by gathering the signal energy matched to its template, and, at the end of each symbol time, presenting the result as a peak amplitude. In general, a digital communications receiver processes received signals with both kinds of filters. The task of the conventional filter is to isolate and extract a high-fidelity estimate of the signal for presentation to the matched filter. The matched filter gathers the received signal energy, and when its output is sampled (at $t = T$), a voltage proportional to that energy is produced for subsequent detection and post-detection processing.

3.3.3.2 Nyquist Pulse and Square-Root Nyquist Pulse

Consider a sequence of data impulses at a transmitter input compared with the resulting sequence of pulses out of a raised-cosine matched filter (before sampling). In Figure 3.21, transmitted data is represented by impulse waveforms that occur at times $\tau_0, \tau_1, \ldots$ Filtering spreads the input waveforms, and thus delays them in time. We use the notation, $t_0, t_1, \ldots$, to denote received time. The impulse event that was transmitted at time τ_0 arrives at the receiver at time t_0 corresponding to the start of the output pulse event. The premainlobe tail of a demodulated pulse is referred to as its *precursor.* For a real system with a fixed system-time reference, causality dictates that $t_0 \geq \tau_0$, and the time difference between τ_0 and t_0 represents any propagation delay in the system. In this example, the time duration from the start of a demodulated pulse precursor until the appearance of its mainlobe or peak amplitude is $3T$ (three pulse-time durations). Each output pulse in the sequence is superimposed with other pulses; each pulse has an effect on the main lobes of three earlier and three later pulses. When a pulse is filtered (shaped) so that it occupies more than one symbol time, we define the pulse *support time* as the total number of symbol intervals over which the pulse persists. In Figure 3.21, the pulse support time consists of 6-symbol intervals (7 data points with 6 intervals between them).

The impulse response of a root-raised cosine filter, called the *square-root Nyquist pulse,* is shown in Figure 3.22a (normalized to a peak value of unity, with a filter rolloff of $r = 0.5$). The impulse response of the raised-cosine filter, called the

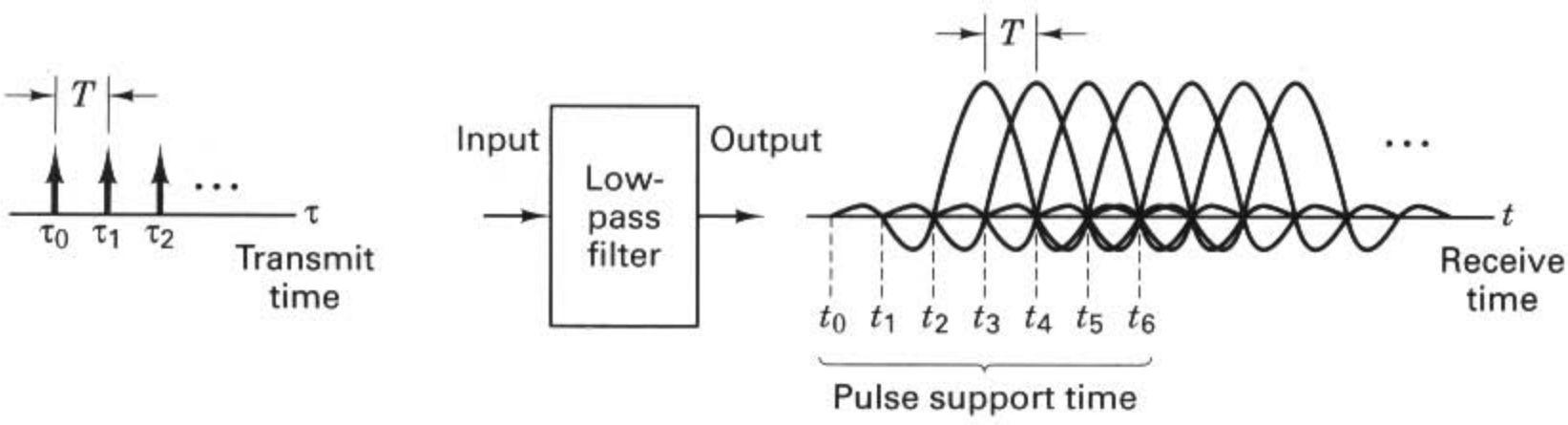


Figure 3.21 Filtered impulse sequence: output versus input.

Nyquist pulse, is shown in Figure 3.22b (with the same normalization and filter rolloff). Inspecting these two pulse shapes, we see that they have a similar appearance, but the square-root Nyquist pulse makes slightly faster transitions, thus its spectrum (root-raised cosine) does not decay as rapidly as the spectrum (raised cosine) of the Nyquist pulse. Another subtle but important difference is that the square-root Nyquist pulse *does not* exhibit zero ISI (you can verify that the pulse tails in Figure 3.22a do not go through zero amplitude at the symbol times). However, if a root-raised cosine filter is used at both the transmitter and the receiver, the product of these transfer functions being a raised cosine, will give rise to an output having zero ISI.

It is interesting to see how the square-root Nyquist pulses appear at the output of a transmitter and how they appear after demodulation with a root-raised cosine MF. Figure 3.23a illustrates an example of sending a sequence of message symbols {+1 +1 −1 +3 +1 +3} from a 4-ary set, where the members of the alphabet set are: {±1, ±3}. Consider that the pulse modulation is 4-ary PAM, and that the pulses have been shaped with a root-raised cosine filter, having a roll-off value of 0.5. The analog waveform in this figure represents the transmitter output. Since the output waveform from any filter is delayed in time, then in Figure 3.23a, the input message symbols (shown as approximate impulses) have been delayed the same amount as the output waveform in order to align the message sequence with its corresponding filtered waveform (the square-root Nyquist shaped-pulse sequence). This is just a visual convenience so that the reader can compare the filter input with its output. It is, of course, only the output analog waveform that is transmitted (or modulated) onto a carrier wave.

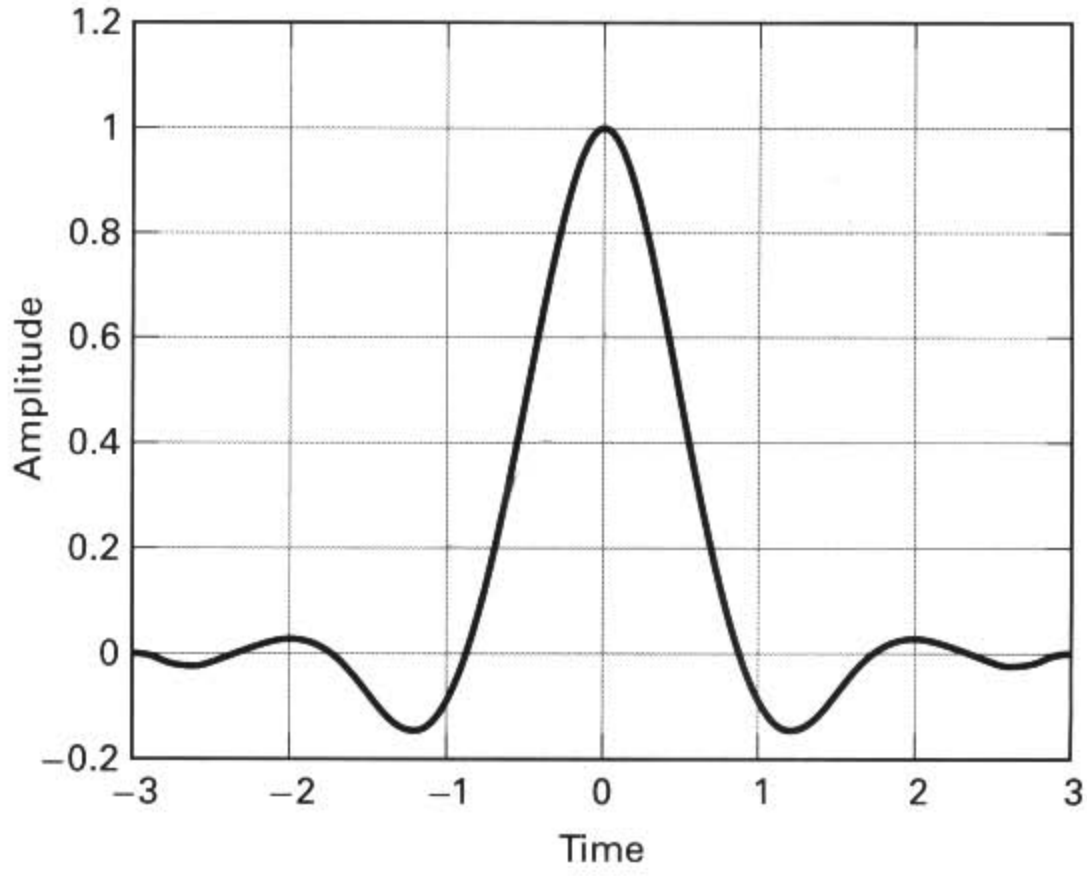


Figure 3.22a Square-root Nyquist pulse.

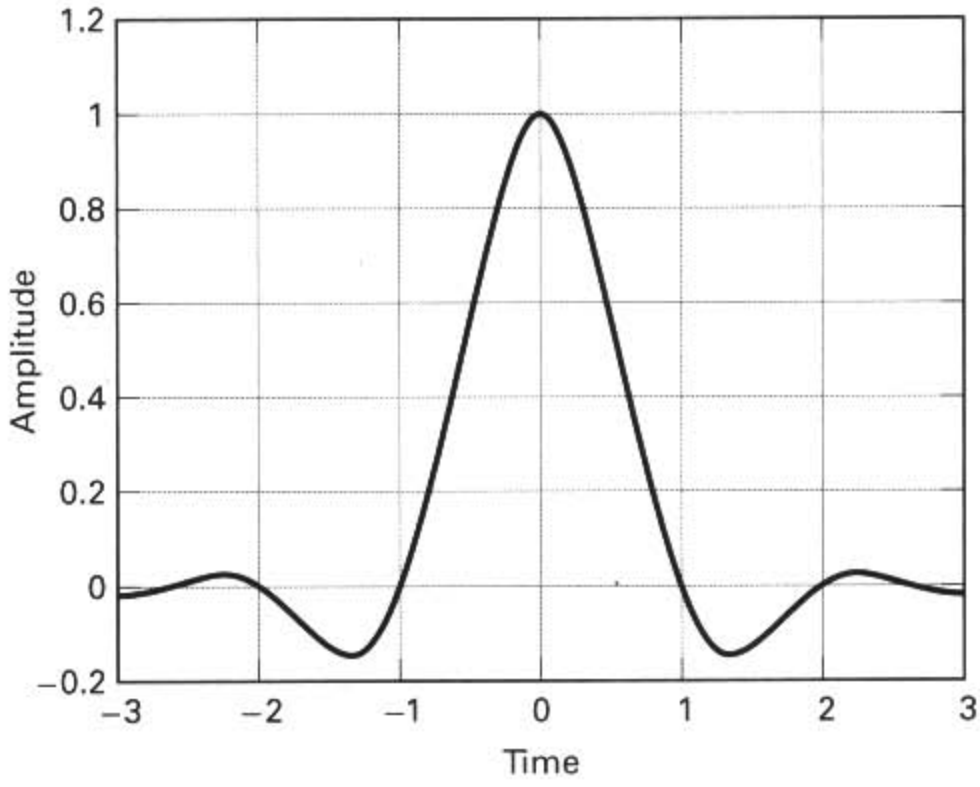


Figure 3.22b Nyquist pulse.

Figure 3.23b shows the same delayed message samples together with the output waveform from the root-raised cosine MF, yielding a raised-cosine transfer function for the overall system. Let us describe a simple test to determine if the filtered output (assuming no noise) contains ISI. It is only necessary to sample the filtered waveform at the times corresponding to the original input samples; if

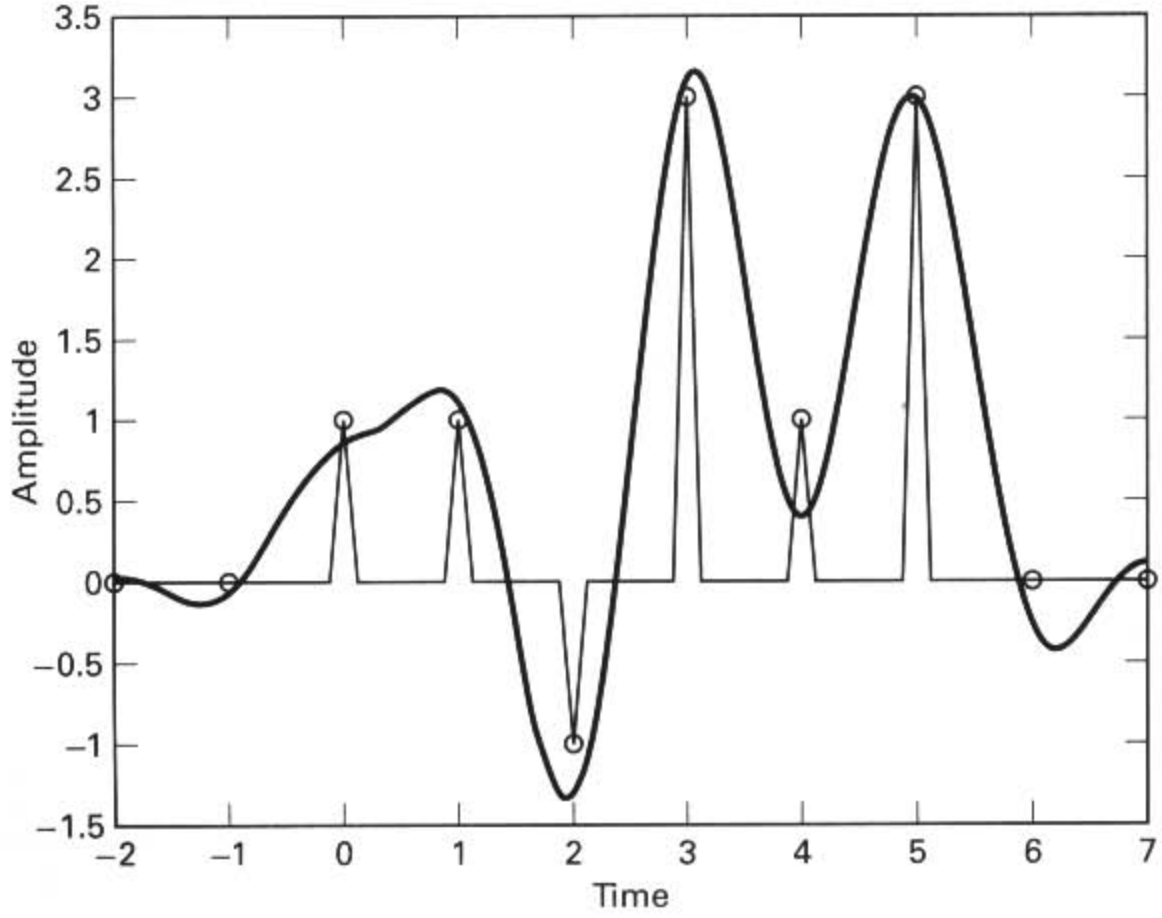


Figure 3.23a Square-root Nyquist-shaped *M*-ary waveform and delayed-input sample values.

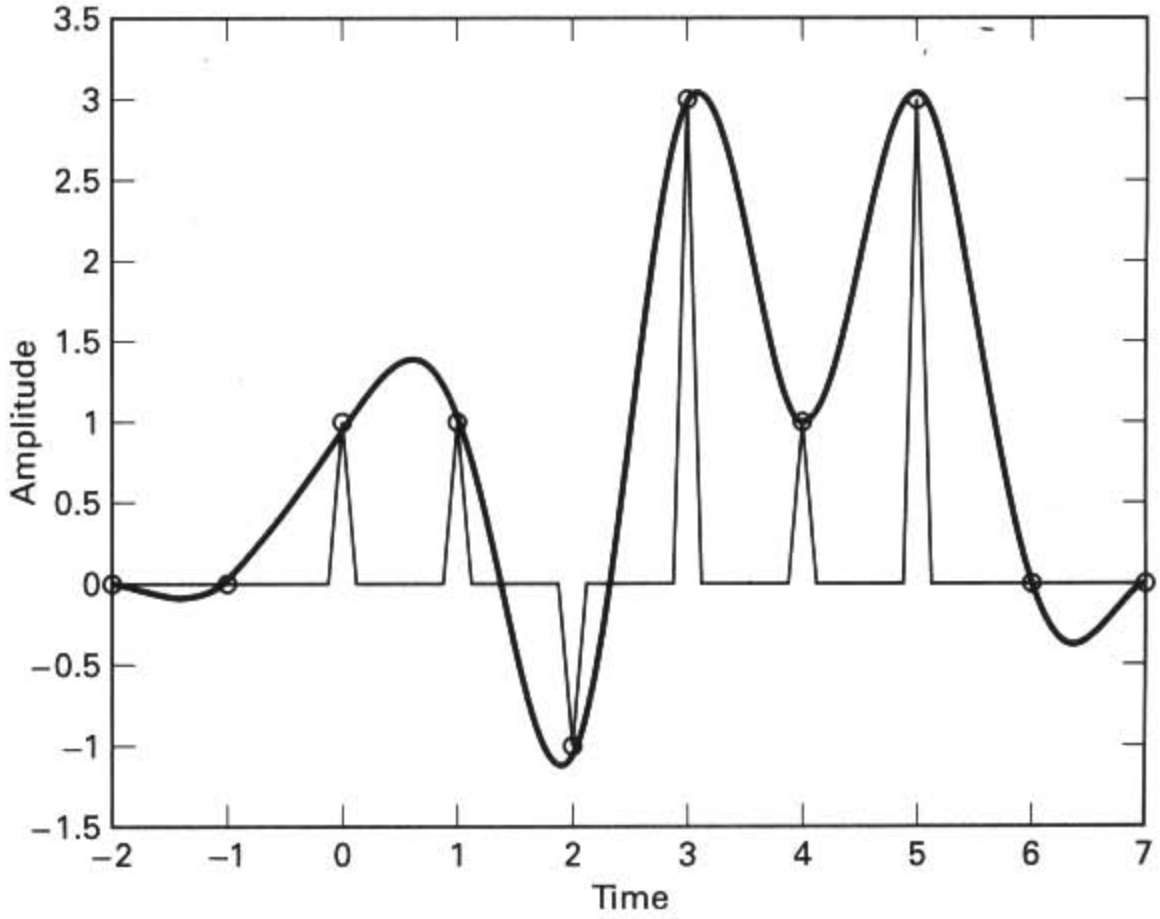


Figure 3.23b Output of raised-cosine matched filter and delayed-input sample values.

the resulting sample values are unchanged from those of the original message, then the filter output has zero ISI (at the sample times). When Figures 3.23a and 3.23b are compared with regard to ISI, it should be apparent that sampling the square-root Nyquist waveform of Figure 3.23a (transmitter output) *will not* yield the exact original samples; however, sampling the Nyquist waveform in Figure 3.23b (MF output) *will* yield the exact original samples. This supports the statement that a Nyquist filter yields zero ISI at the sample points, while any other filter does not do so.

3.4 EQUALIZATION

3.4.1 Channel Characterization

Many communication channels (e.g., telephone, wireless) can be characterized as band-limited linear filters with an impulse response $h_c(t)$ and a frequency response

$$H_c(f) = |H_c(f)|e^{j\theta_c(f)} \tag{3.82}$$

where $h_c(t)$ and $H_c(f)$ are Fourier transform pairs, $|H_c(f)|$ is the channel's amplitude response, and $\theta_c(f)$ is the channel's phase response. In order to achieve ideal (nondistorting) transmission characteristics over a channel, it was shown in Section 1.6.3, that within a signal's bandwidth W, $|H_c(f)|$ *must be constant.* Also, $\theta_c(f)$ *must be a linear function of frequency,* which is tantamount to saying that the delay must be constant for all spectral components of the signal. If $|H_c(f)|$ is not constant

within W, then the effect is amplitude distortion. If $\theta_c(f)$ is not a linear function of frequency within W, then the effect is phase distortion. For many channels that exhibit distortion of this type, such as fading channels, amplitude and phase distortion typically occur together. For a transmitted sequence of pulses, such distortion manifests itself as a signal dispersion or "smearing" so that any one pulse in the received demodulated sequence is not well defined. The overlap or smearing, known as *intersymbol interference* (ISI), described in Section 3.3, arises in most modulation systems; it is one of the major obstacles to reliable high-speed data transmission over bandlimited channels. In the broad sense, the name "equalization" refers to any signal processing or filtering technique that is designed to eliminate or reduce ISI.

In Figure 2.1, equalization is partitioned into two broad categories. The first category, *maximum-likelihood sequence estimation* (MLSE), entails making measurements of $h_c(t)$ and then providing a means for adjusting the receiver to the transmission environment. The goal of such adjustments is to enable the detector to make good estimates from the demodulated distorted pulse sequence. With an MLSE receiver, the distorted samples are not reshaped or directly compensated in any way; instead, the mitigating technique for the MLSE receiver is to adjust itself in such a way that it can better deal with the distorted samples. (An example of this method, known as Viterbi equalization, is treated in Section 15.7.1.) The second category, *equalization with filters,* uses filters to compensate the distorted pulses. In this second category, the detector is presented with a sequence of demodulated samples that the equalizer has modified or "cleaned up" from the effects of ISI. Equalizing with filters, the more popular approach and the one described in this section, lends itself to further partitioning. The filters can be described as to whether they are linear devices that contain only feedforward elements *transversal equalizers*), or whether they are nonlinear devices that contain both feedforward and feedback elements (*decision feedback equalizers*). They can be grouped according to the automatic nature of their operation, which may be either *preset* or *adaptive.* They also can be grouped according to the filter's resolution or update rate. Are predetection samples provided only on symbol boundaries, that is, one sample per symbol? If so, the condition is known as *symbol spaced.* Are multiple samples provided for each symbol? If so, this condition is known as *fractionally spaced.*

We now modify Equation (3.77) by letting the receiving/equalizing filter be replaced by a separate receiving filter and equalizing filter, defined by frequency transfer functions $H_r(f)$ and $H_e(f)$, respectively. Also, let the overall system transfer function $H(f)$ be a raised-cosine filter, designated $H_{RC}(f)$. Thus, we write

$$H_{RC}(f) = H_t(f)\, H_c(f)\, H_r(f)\, H_e(f) \tag{3.83}$$

In practical systems, the channel's frequency transfer function $H_c(f)$ and its impulse response $h_c(t)$ are not known with sufficient precision to allow for a receiver design to yield zero ISI for all time. Usually, the transmit and receive filters are chosen to be matched so that

$$H_{RC}(f) = H_t(f)\, H_r(f) \tag{3.84}$$

In this way, $H_t(f)$ and $H_r(f)$ each have frequency transfer functions that are the square root of the raised cosine (root-raised cosine). Then, the equalizer transfer function needed to compensate for channel distortion is simply the inverse of the channel transfer function:

$$H_e(f) = \frac{1}{H_c(f)} = \frac{1}{|H_c(f)|} e^{-j\theta_c(f)} \tag{3.85}$$

Sometimes a system frequency transfer function manifesting ISI at the sampling points is purposely chosen (e.g., a Gaussian filter transfer function). The motivation for such a transfer function is to improve bandwidth efficiency, compared with using a raised-cosine filter. When such a design choice is made, the role of the equalizing filter is not only to compensate for the channel-induced ISI, but also to compensate for the ISI brought about by the transmitter and receiver filters [7].

3.4.2 Eye Pattern

An eye pattern is the display that results from measuring a system's response to baseband signals in a prescribed way. On the vertical plates of an oscilloscope we connect the receiver's response to a random pulse sequence. On the horizontal plates we connect a sawtooth wave at the signaling frequency. In other words, the horizontal time base of the oscilloscope is set equal to the symbol (pulse) duration. This setup superimposes the waveform in each signaling interval into a family of traces in a single interval $(0, T)$. Figure 3.24 illustrates the eye pattern that results for binary antipodal (bipolar pulse) signaling. Because the symbols stem from a random source, they are sometimes positive and sometimes negative, and the persistence of the cathode ray tube display allows us to see the resulting pattern shaped as an eye. The width of the opening indicates the time over which sampling for detection might be performed. Of course, the optimum sampling time corresponds to the maximum eye opening, yielding the greatest protection against noise.

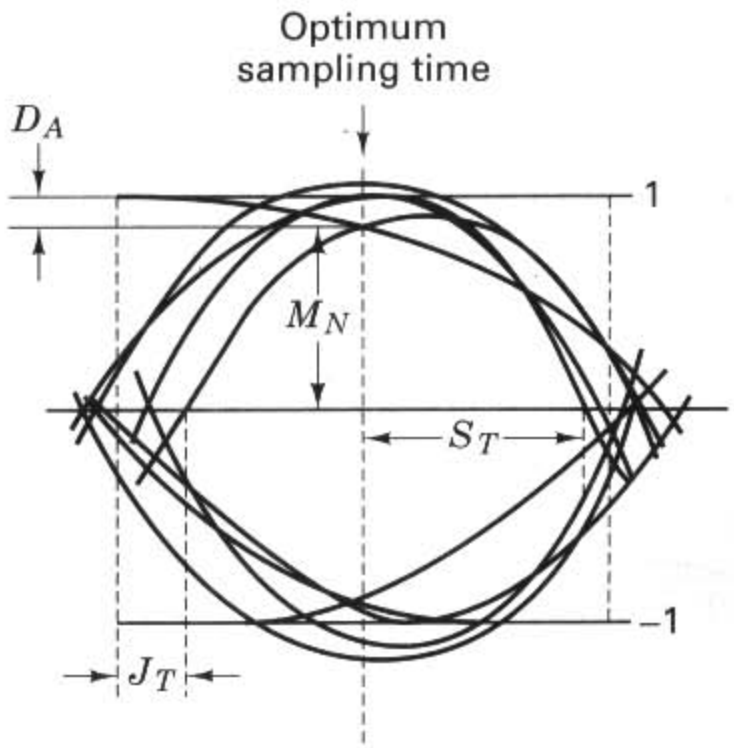


Figure 3.24 Eye Pattern.

If there were no filtering in the system—that is, if the bandwidth corresponding to the transmission of these data pulses were infinite—then the system response would yield ideal rectangular pulse shapes. In that case, the pattern would look like a box rather than an eye. In Figure 3.24, the range of amplitude differences labelled D_A is a measure of distortion caused by ISI, and the range of time differences of the zero crossings labelled J_T is a measure of the timing jitter. Measures of noise margin M_N and sensitivity-to-timing error S_T are also shown in the figure. In general, the most frequent use of the eye pattern is for qualitatively assessing the extent of the ISI. As the eye closes, ISI is increasing; as the eye opens, ISI is decreasing.

3.4.3 Equalizer Filter Types

3.4.3.1 Transversal Equalizer

A training sequence used for equalization is often chosen to be a noise-like sequence, "rich" in spectral content, which is needed to estimate the channel frequency response. In the simplest sense, training might consist of sending a single narrow pulse (approximately an ideal impulse) and thereby learning the impulse response of the channel. In practice, a pseudonoise (PN) signal is preferred over a single pulse for the training sequence because the PN signal has larger average power and hence larger SNR for the same peak transmitted power. For describing the transversal filter, consider that a single pulse was transmitted over a system designated to have a raised-cosine transfer function $H_{\text{RC}}(f) = H_t(f)\, H_r(f)$. Also consider that the channel induces ISI, so that the received demodulated pulse exhibits distortion, as shown in Figure 3.25, such that the pulse sidelobes do not go through zero at sample times adjacent to the mainlobe of the pulse. The distortion can be viewed as positive or negative echoes occurring both before and after the mainlobe. To achieve the desired raised-cosine transfer function, the equalizing filter should have a frequency response $H_e(f)$, as shown in Equation (3.85), such that the actual channel response when multiplied by $H_e(f)$ yields $H_{\text{RC}}(f)$. In other words, we would like the equalizing filter to generate a set of canceling echoes.

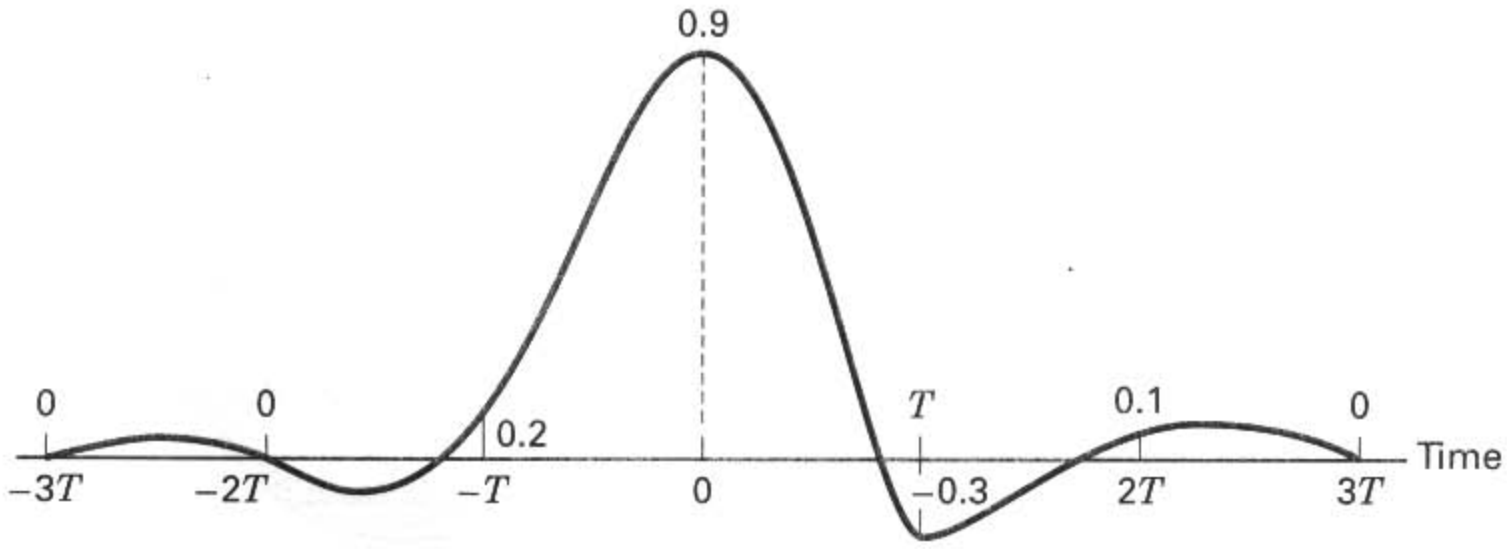


Figure 3.25 Received pulse exhibiting distortion.

Since we are interested in sampling the equalized waveform at only a few predetermined sampling times, then the design of an equalizing filter can be a straightforward task.

The transversal filter, depicted in Figure 3.26, is the most popular form of an easily adjustable equalizing filter consisting of a delay line with T-second taps (where T is the symbol duration). In such an equalizer, the current and past values of the received signal are linearly weighted with equalizer coefficients or tap weights $\{c_n\}$ and are then summed to produce the output. The main contribution is from a central tap, with the other taps contributing echoes of the main signal at symbol intervals on either side of the main signal. If it were possible for the filter to have an infinite number of taps, then the tap weights could be chosen to force the system impulse response to zero at all but one of the sampling times, thus making $H_e(f)$ correspond exactly to the inverse of the channel transfer function in Equation (3.85). Even though an infinite length filter is not realizable, one can still specify practical filters that approximate the ideal case.

In Figure 3.26, the outputs of the weighted taps are amplified, summed, and fed to a decision device. The tap weights $\{c_n\}$ need to be chosen so as to subtract the effects of interference from symbols adjacent in time to the desired symbol. Consider that there are $(2N + 1)$ taps with weights $c_{-N}, c_{-N+1}, \ldots, c_N$. Output samples $\{z(k)\}$ of the equalizer are then found by convolving the input samples $\{x(k)\}$ and tap weights $\{c_n\}$ as follows:

$$z(k) = \sum_{n=-N}^{N} x(k-n)\, c_n \quad k = -2N, \ldots, 2N \quad n = -N, \ldots, N \tag{3.86}$$

where $k = 0, \pm 1, \pm 2, \ldots$ is a time index that is shown in parentheses. (Time may take on any range of values.) The index n is used two ways—as a time offset, and as a filter coefficient identifier (which is an address in the filter). When used in the latter

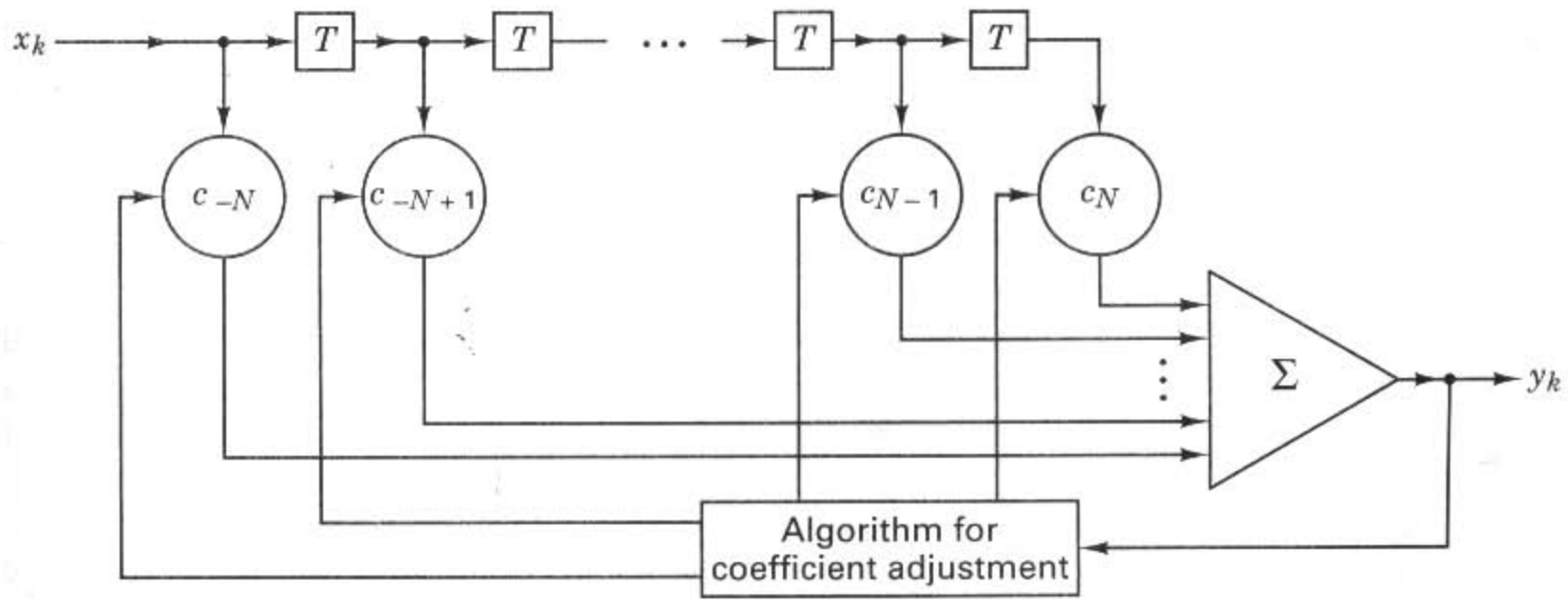


Figure 3.26 Transversal filter.

sense, it is shown as a subscript. By defining the vectors **z** and **c** and the matrix **x** as, respectively,

$$\mathbf{z} = \begin{bmatrix} z(-2N) \\ \vdots \\ z(0) \\ \vdots \\ z(2N) \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} c_{-N} \\ \vdots \\ c_0 \\ \vdots \\ c_N \end{bmatrix} \tag{3.87}$$

and

$$\mathbf{x} = \begin{bmatrix} x(-N) & 0 & 0 & \cdots & 0 & 0 \\ x(-N+1) & x(-N) & 0 & \cdots & \cdots & \cdots \\ \vdots & & & \vdots & & \vdots \\ x(N) & x(N-1) & x(N-2) & \cdots & x(-N+1) & x(-N) \\ \vdots & & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x(N) & x(N-1) \\ 0 & 0 & 0 & \cdots & 0 & x(N) \end{bmatrix} \tag{3.88}$$

we can describe the relationship among $\{z(k)\}$, $\{x(k)\}$, and $\{c_n\}$ more compactly as

$$\mathbf{z} = \mathbf{x}\,\mathbf{c} \tag{3.89a}$$

And whenever the matrix **x** is square, with its rows and columns each having the same dimension as the number of elements in **c,** we can find **c** by solving the following equation:

$$\mathbf{c} = \mathbf{x}^{-1}\mathbf{z} \tag{3.89b}$$

Notice that the size of the vector **z** and the number of rows in the matrix **x** may be chosen to be any value, because one might be interested in the ISI at sample points far removed from the mainlobe of the pulse in question. In Equations (3.86) through (3.88), the index k was arbitrarily chosen to allow for $4N + 1$ sample points. The vectors **z** and **c** have dimensions $4N + 1$ and $2N + 1$, respectively, and the matrix **x** is nonsquare with dimensions $4N + 1$ by $2N + 1$. Such equations are referred to as an overdetermined set (i.e., there are more equations than unknowns). One can solve such a problem in a deterministic way known as the *zero-forcing* solution, or, in a statistical way, known as the *minimum mean-square error* (MSE) solution.

Zero-Forcing Solution. This solution starts by disposing of the top N and bottom N rows of the matrix **x** in Equation (3.88), thereby transforming **x** into a square matrix of dimension $2N + 1$ by $2N + 1$, transforming **z** into a vector of dimension $2N + 1$, and yielding in Equation (3.89a), a deterministic set of $2N + 1$ simultaneous equations. This zero-forcing solution minimizes the peak ISI distortion by selecting the $\{c_n\}$ weights so that the equalizer output is forced to zero at N sample points on either side of the desired pulse. In other words, the weights are chosen so that

$$z(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k = \pm 1, \pm 2, \ldots, \pm N \end{cases} \tag{3.90}$$

Equation (3.89) is used to solve the $2N + 1$ simultaneous equations for the set of $2N + 1$ weights $\{c_n\}$. The required length of the filter (number of tap weights) is a function of how much smearing the channel may introduce. For such an equalizer with finite length, the peak distortion is guaranteed to be minimized only if the eye pattern is initially open. However, for high-speed transmission and channels introducing much ISI, the eye is often closed before equalization [8]. Since the zero-forcing equalizer neglects the effect of noise, it is not always the best system solution.

Example 3.5 A Zero-Forcing Three-Tap Equalizer

Consider that the tap weights of an equalizing transversal filter are to be determined by transmitting a single impulse as a training signal. Let the equalizer circuit in Figure 3.26 be made up of just three taps. Given a received distorted set of pulse samples $\{x(k)\}$, with voltage values 0.0, 0.2, 0.9, –0.3, 0.1, as shown in Figure 3.25, use a zero-forcing solution to find the weights $\{c_{-1}, c_0, c_1\}$ that reduce the ISI so that the equalized pulse samples $\{z(k)\}$ have the values, $\{z(-1) = 0,\ z(0) = 1,\ z(1) = 0\}$. Using these weights, calculate the ISI values of the equalized pulse at the sample times $k = \pm 2, \pm 3$. What is the largest magnitude sample contributing to ISI, and what is the sum of all the ISI magnitudes?

Solution

For the channel impulse response specified, Equation (3.89) yields

$$\mathbf{z} = \mathbf{x}\,\mathbf{c}$$

or

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x(0) & x(-1) & x(-2) \\ x(1) & x(0) & x(-1) \\ x(2) & x(1) & x(0) \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 & 0.2 & 0 \\ -0.3 & 0.9 & 0.2 \\ 0.1 & -0.3 & 0.9 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix}$$

Solving these three simultaneous equations results in the following weights:

$$\begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} -0.2140 \\ 0.9631 \\ 0.3448 \end{bmatrix}$$

The values of the equalized pulse samples $\{z(k)\}$ corresponding to sample times $k =$ –3, –2, –1, 0, 1, 2, 3 are computed by using the preceding weights in Equation (3.89a), yielding

The sample of greatest magnitude contributing to ISI equals 0.0428, and the sum of all

The values of the equalized pulse samples $\{z(k)\}$ corresponding to sample times $k = -3, -2, -1, 0, 1, 2, 3$ are computed by using the preceding weights in Equation (3.89a), yielding

$$0.0000,\ -0.0428,\ 0.0000,\ 1.0000,\ 0.0000,\ -0.0071,\ 0.0345$$

The sample of greatest magnitude contributing to ISI equals 0.0428, and the sum of all the ISI magnitudes equals 0.0844. It should be clear that this three-tap equalizer has forced the sample points on either side of the equalized pulse to be zero. If the equalizer is made longer than three taps, more of the equalized sample points can be forced to a zero value.

Minimum MSE Solution. A more robust equalizer is obtained if the $\{c_n\}$ tap weights are chosen to minimize the mean-square error (MSE) of all the ISI terms plus the noise power at the output of the equalizer [9]. MSE is defined as the expected value of the squared difference between the desired data symbol and the estimated data symbol. One can use the set of overdetermined equations to obtain a minimum MSE solution by multiplying both sides of Equation (3.89a) by $\mathbf{x}^T$, which yields [10]

$$\mathbf{x}^T\mathbf{z} = \mathbf{x}^T\mathbf{x}\mathbf{c} \tag{3.91a}$$

and

$$\mathbf{R}_{xz} = \mathbf{R}_{xx}\mathbf{c} \tag{3.91b}$$

where $\mathbf{R}_{xz} = \mathbf{x}^T\mathbf{z}$ is called the *cross-correlation* vector and $\mathbf{R}_{xx} = \mathbf{x}^T\mathbf{x}$ is called the *autocorrelation* matrix of the input noisy signal. In practice, $\mathbf{R}_{xz}$ and $\mathbf{R}_{xx}$ are unknown *a priori,* but can be approximated by transmitting a test signal over the channel and using time average estimates to solve for the tap weights from Equation (3.91), as follows:

$$\mathbf{c} = \mathbf{R}_{xx}^{-1}\mathbf{R}_{xz} \tag{3.92}$$

In the case of the deterministic zero-forcing solution, the $\mathbf{x}$ matrix must be square. But to achieve the minimum MSE (statistical) solution, one starts with an overdetermined set of equations and hence a *nonsquare* $\mathbf{x}$ matrix, which then gets transformed to a *square* autocorrelation matrix $\mathbf{R}_{xx} = \mathbf{x}^T\mathbf{x}$, yielding a set of $2N + 1$ simultaneous equations, whose solution leads to tap weights that minimize the MSE. The size of the vector $\mathbf{c}$ and the number of columns of the matrix $\mathbf{x}$ correspond to the number of taps in the equalizing filter. Most high-speed telephone-line modems use an MSE weight criterion because it is superior to a zero-forcing criterion; it is more robust in the presence of noise and large ISI [8].

Example 3.6 A Minimum MSE 7-Tap Equalizer

Consider that the tap weights of an equalizing transversal filter are to be determined by transmitting a single impulse as a training signal. Let the equalizer circuit in Figure 3.26 be made up of seven taps. Given a received distorted set of pulse samples $\{x(k)\}$, with values 0.0108, –0.0558, 0.1617, 1.0000, –0.1749, 0.0227, 0.0110, use a minimum MSE solution to find the value of the weights $\{c_n\}$ that will minimize the ISI. With these weights, calculate the resulting values of the equalized pulse samples at the fol-

lowing times: $\{k = 0, \pm1, \pm2, \ldots, \pm6\}$. What is the largest magnitude sample contributing to ISI, and what is the sum of all the ISI magnitudes?

Solution

For a seven-tap filter ($N = 3$), one can form the **x** matrix in Equation (3.88) that has dimensions $4N + 1$ by $2N + 1 = 13 \times 7$:

$$\mathbf{x} = \begin{bmatrix} 0.0110 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0227 & 0.0110 & 0 & 0 & 0 & 0 & 0 \\ -0.1749 & 0.0227 & 0.0110 & 0 & 0 & 0 & 0 \\ 1.0000 & -0.1749 & 0.0227 & 0.0110 & 0 & 0 & 0 \\ 0.1617 & 1.0000 & -0.1749 & 0.0227 & 0.0110 & 0 & 0 \\ -0.0558 & 0.1617 & 1.0000 & -0.1749 & 0.0227 & 0.0110 & 0 \\ 0.0108 & -0.0558 & 0.1617 & 1.0000 & -0.1749 & 0.0227 & 0.0110 \\ 0 & 0.0108 & -0.0558 & 0.1617 & 1.0000 & -0.1749 & 0.0227 \\ 0 & 0 & 0.0108 & -0.0558 & 0.1617 & 1.0000 & -0.1749 \\ 0 & 0 & 0 & 0.0108 & -0.0558 & 0.1617 & 1.0000 \\ 0 & 0 & 0 & 0 & 0.0108 & -0.0558 & 0.1617 \\ 0 & 0 & 0 & 0 & 0 & 0.0108 & -0.0558 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0108 \end{bmatrix}$$

Using this **x** matrix, one can form the autocorrelation matrix $\mathbf{R}_{xx}$ and the cross-correlation vector $\mathbf{R}_{xz}$, defined in Equation (3.91). With the help of a computer to invert $\mathbf{R}_{xx}$ and perform matrix multiplication, the solution for the tap weights $\{c_{-3}, c_{-2}, c_{-1}, c_0, c_1, c_2, c_3\}$ shown in Equation (3.92) yields

$$-0.0116,\ 0.0108,\ 0.1659,\ 0.9495,\ -0.1318,\ 0.0670,\ -0.0269$$

Using these weights in Equation (3.89a), we solve for the 13 equalized samples $\{z(k)\}$ at times $k = -6, -5, \ldots, 5, 6$:

$$-0.0001,\ -0.0001,\ \ 0.0041,\ 0.0007,\ \ 0.0000,\ -0.0000,\ 1.0000,\ 0.0003,$$
$$-0.0007,\ \ 0.0015,\ -0.0095,\ 0.0022,\ -0.0003$$

The largest magnitude sample contributing to ISI equals 0.0095, and the sum of all the ISI magnitudes equals 0.0195.

3.4.3.2 Decision Feedback Equalizer

The basic limitation of a linear equalizer, such as the transversal filter, is that it performs poorly on channels having spectral nulls [11]. Such channels are often encountered in mobile radio applications. A decision feedback equalizer (DFE) is a nonlinear equalizer that uses previous detector decisions to eliminate the ISI on pulses that are currently being demodulated. The ISI being removed was caused by the tails of previous pulses; in effect, the distortion on a current pulse that was caused by previous pulses is subtracted.

Figure 3.27 shows a simplified block diagram of a DFE where the forward filter and the feedback filter can each be a linear filter, such as a transversal filter. The figure also illustrates how the filter tap weights are updated adaptively. (See the following section.) The nonlinearity of the DFE stems from the nonlinear

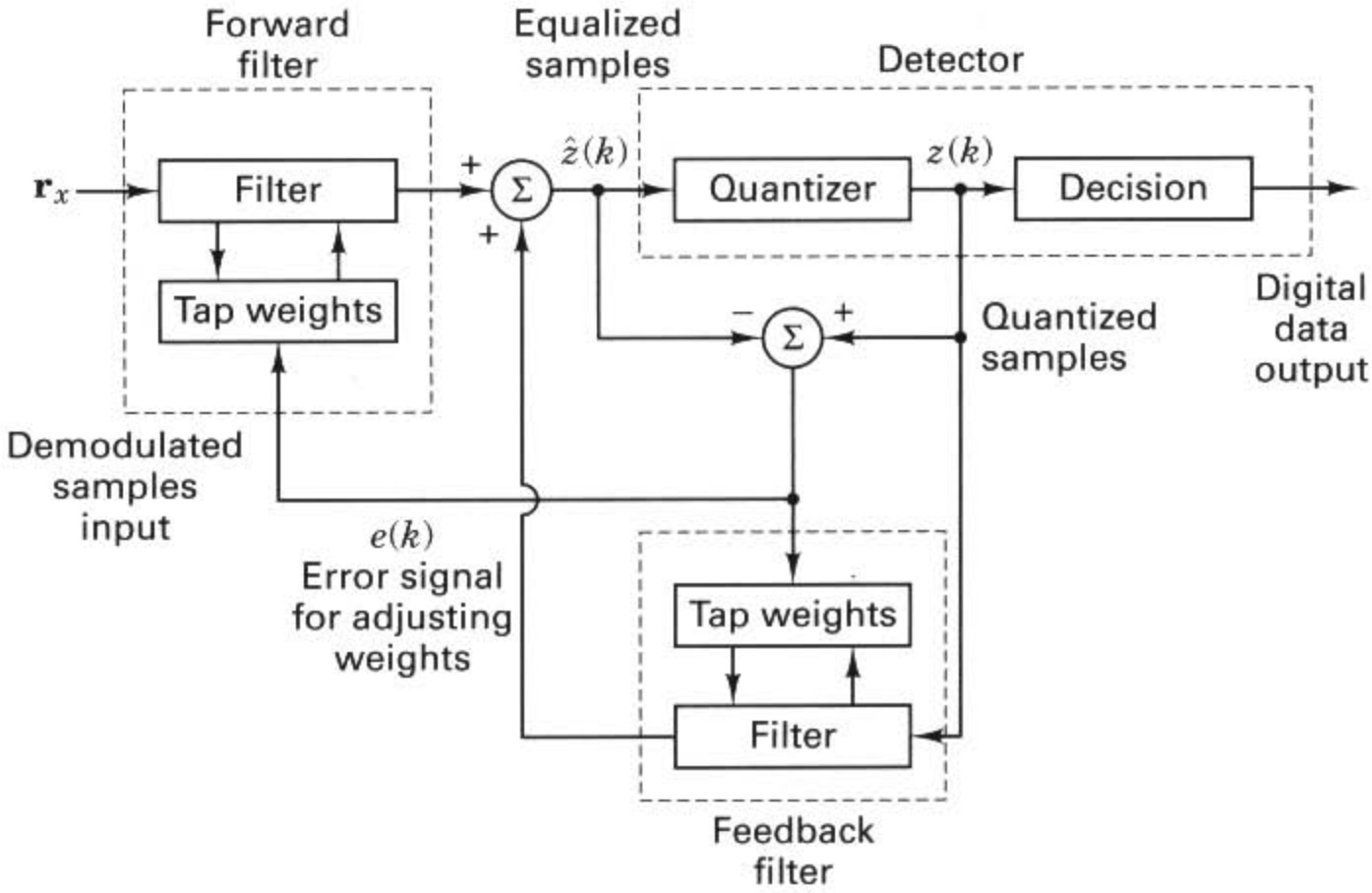


Figure 3.27 Decision Feedback Equalizer.

characteristic of the detector that provides an input to the feedback filter. The basic idea of a DFE is that if the values of the symbols previously detected are known (past decisions are assumed to be correct), then the ISI contributed by these symbols can be canceled out exactly at the output of the forward filter by subtracting past symbol values with appropriate weighting. The forward and feedback tap weights can be adjusted simultaneously to fulfill a criterion such as minimizing the MSE.

When only a forward filter is used, the output of the filter contains channel noise contributed from every sample in the filter. The advantage of a DFE implementation is that the feedback filter, which is additionally working to remove ISI, operates on noiseless quantized levels, and thus its output is free of channel noise.

3.4.4 Preset and Adaptive Equalization

On channels whose frequency responses are known and time invariant, the channel characteristics can be measured and the filter's tap weights adjusted accordingly. If the weights remain fixed during transmission of data, the equalization is called *preset* equalization; one very simple method of preset equalization consists of setting the tap weights $\{c_n\}$ according to some average knowledge of the channel. This was used for data transmission over voice-grade telephone lines at less than 2400 bit/s. Another preset method consists of transmitting a training sequence that is compared at the receiver with a locally generated sequence. The differences between the two sequences are used to set $\{c_n\}$. The significant aspect of any preset method

is that it is done once at the start of transmission or seldom (when transmission is broken and needs to be reestablished).

Another type of equalization, capable of tracking a slowly time-varying channel response, is known as *adaptive* equalization. It can be implemented to perform tap-weight adjustments periodically or continually. Periodic adjustments are accomplished by periodically transmitting a preamble or short training sequence of digital data that is known in advance by the receiver. The receiver also uses the preamble to detect start of transmission, to set the automatic gain control (AGC) level, and to align internal clocks and local oscillator with the received signal. Continual adjustments are accomplished by replacing the known training sequence with a sequence of data symbols estimated from the equalizer output and treated as known data. When performed continually and automatically in this way, the adaptive procedure (the most popular) is referred to as *decision directed* [11]. The name "decision directed" is not to be confused with decision feedback (DFE). Decision directed only addresses how filter tap weights are adjusted—that is, with the help of a signal from the detector. DFE, however, refers to the fact that there exists an additional filter that operates on the detector output and recursively feeds back a signal to the detector input. Thus, with DFE there are two filters, a feed-forward filter and a feedback filter that process the data and help mitigate the ISI.

A disadvantage of preset equalization is that it requires an initial training period that must be invoked at the start of any new transmission. Also, a time-varying channel can degrade system performance due to ISI, since the tap weights are fixed. Adaptive equalization, particularly *decision-directed* adaptive equalization, successfully cancels ISI when the initial probability of error due to probability of error exceeds one percent, (rule of thumb). If the probability of error exceeds one percent, the decision directed equalizer might not converge. A common solution to this problem is to initialize the equalizer with an alternate process, such as a preamble to provide good channel-error performance, and then switch to the decision-directed mode. To avoid the overhead represented by a preamble, many systems designed to operate in a continuous broadcast mode use *blind equalization* algorithms to form initial channel estimates. These algorithms adjust filter coefficients in response to sample statistics rather than in response to sample decisions [11].

Automatic equalizers use iterative techniques to estimate the optimum coefficients. The simultaneous equations described in Equation (3.89) do not include the affects of channel noise. To obtain a stable solution to the filter weights, it is necessary that the data be averaged to obtain stable signal statistics, or the noisy solutions obtained from the noisy data must be averaged. Considerations of algorithm complexity and numerical stability most often lead to algorithms that average noisy solutions. The most robust of this class of algorithm is the least-mean-square (LMS) algorithm. Each iteration of this algorithm uses a noisy estimate of the error *gradient* to adjust the weights in the direction to reduce the average mean-square error. The noisy gradient is simply the product $e(k)\ \mathbf{r}_x$ of an error scalar $e(k)$ and the data vector $\mathbf{r}_x$. The vector $\mathbf{r}_x$ is the vector of noise-corrupted channel samples residing in the equalizer filter at time k. Earlier, an impulse was transmitted and the equalizing filter operated on a sequence of samples (a vector) that represented

the impulse response of the channel. We displayed these received samples (in time-shifted fashion) as the matrix **x.** Now, rather than dealing with the response to an impulse, consider that data is sent and thus the vector of received samples $\mathbf{r}_x$ at the input to the filter (Figure 3.27) represents the data response of the channel. The error is formed as the difference between the desired output signal and the filter output signal and is given by

$$e(k) = z(k) - \hat{z}(k) \tag{3.93}$$

where $z(k)$ is the desired output signal (a sample free of ISI) and $\hat{z}(k)$ is an estimate of $z(k)$ at time k out of the filter (into the quantizer of Figure 3.27), which is obtained as follows:

$$\hat{z}(k) = \mathbf{c}^T\mathbf{r}_x = \sum_{n=-N}^{N} x(k-n)c_n \tag{3.94}$$

In Equation (3.94), the summation represents a convolution of the input data samples with the $\{c_n\}$ tap weights, where c_n refers to the nth tap weight at time k, and $\mathbf{c}^T$ is the transpose of the weight vector at time k. We next show the iterative process that updates the set of weights at each time k as

$$\mathbf{c}(k+1) = \mathbf{c}(k) + \Delta e(k)\mathbf{r}_x \tag{3.95}$$

where $\mathbf{c}(k)$ is the vector of filter weights at time k, and Δ is a small term that limits the coefficient step size and thus controls the rate of convergence of the algorithm as well as the variance of the steady state solution. This simple relationship is a consequence of the orthogonality principle that states that the error formed by an optimal solution is orthogonal to the processed data. Since this is a recursive algorithm (in the weights), care must be exercised to assure algorithm stability. Stability is assured if the parameter Δ is smaller than the reciprocal of the energy of the data in the filter. When stable, this algorithm converges in the mean to the optimal solution but exhibits a variance proportional to the parameter Δ. Thus, while we want the convergence parameter Δ to be large for fast convergence but not so large as to be unstable, we also want it to be small enough for low variance. The parameter Δ is usually set to a fixed small amount [12] to obtain a low-variance steady-state tap-weight solution. Schemes exist that permit Δ to change from large values during initial acquisition to small values for stable steady-state solutions [13].

Note that Equations (3.93) through (3.95) are shown in the context of real signals. When the receiver is implemented in quadrature fashion, such that the signals appear as real and imaginary (or inphase and quadrature) ordered pairs, then each line in Figure 3.27 actually consists of two lines, and Equations (3.93) through (3.95) need to be expressed with complex notation. (Such quadrature implementation is treated in greater detail in Sections 4.2.1 and 4.6.)

3.4.5 Filter Update Rate

Equalizer filters are classified by the rate at which the input signal is sampled. A transversal filter with taps spaced T seconds apart, where T is the symbol time, is called a *symbol-spaced* equalizer. The process of sampling the equalizer output at a rate $1/T$ causes aliasing if the signal is not strictly bandlimited to $1/T$ hertz—that is, the signal's spectral components spaced $1/T$ hertz apart are folded over and superimposed. The aliased version of the signal may exhibit spectral nulls [8]. A filter update rate that is greater than the symbol rate helps to mitigate this difficulty. Equalizers using this technique are called *fractionally-spaced* equalizers. With a fractionally spaced equalizer, the filter taps are spaced at

$$T' \leq \frac{T}{(1+r)} \tag{3.96}$$

seconds apart, where r denotes the excess bandwidth. In other words, the received signal bandwidth is

$$W \leq \frac{(1+r)}{T} \tag{3.97}$$

The goal is to choose T' so that the equalizer transfer function $H_e(f)$ becomes sufficiently broad to accommodate the whole signal spectrum. Note that the signal at the output of the equalizer is still sampled at a rate $1/T$, but since the tap weights are spaced T' seconds apart (the equalizer input signal is sampled at a rate $1/T'$), the equalization action operates on the received signal before its frequency components are aliased. Equalizer simulations over voice-grade telephone lines, with $T' = T/2$, confirm that such fractionally-spaced equalizers outperform symbol-spaced equalizers [14].

3.5 CONCLUSION

In this chapter, we described the detection of binary signals plus Gaussian noise in terms of two basic steps. In the first step the received waveform is reduced to a single number $z(T)$, and in the second step a decision is made as to which signal was transmitted, on the basis of comparing $z(T)$ to a threshold. We discussed how to best choose this threshold. We also showed that a linear filter known as a matched filter or correlator is the optimum choice for maximizing the output signal-to-noise ratio and thus minimizing the probability of error.

We defined intersymbol interference (ISI) and explained the importance of Nyquist's work in establishing a theoretical minimum bandwidth for symbol detection without ISI. We partitioned error-performance degradation into two main types. The first is a simple loss in signal-to-noise ratio. The second, resulting from distortion, is a bottoming-out of the error probability versus the E_b/N_0 curve.

Finally, we described equalization techniques that can be used to mitigate the effects of ISI.

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PROBLEMS

3.1. Determine whether or not $s_1(t)$ and $s_2(t)$ are orthogonal over the interval $(-1.5T_2 < t < 1.5T_2)$, where $s_1(t) = \cos(2\pi f_1 t + \phi_1)$, $s_2(t) = \cos(2\pi f_2 t + \phi_2)$, and $f_2 = 1/T_2$ for the following cases.

(a) $f_1 = f_2$ and $\phi_1 = \phi_2$

(b) $f_1 = \frac{1}{3} f_2$ and $\phi_1 = \phi_2$

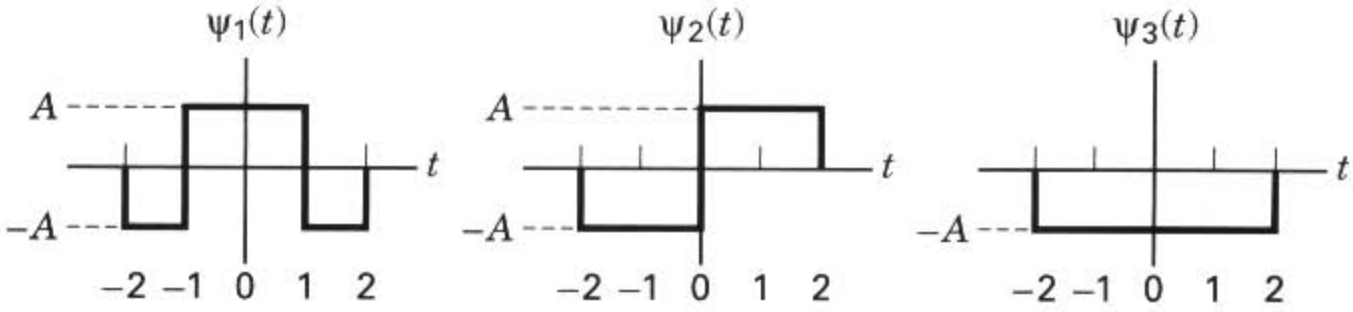

Figure P3.2

(c) $f_1 = 2f_2$ and $\phi_1 = \phi_2$
(d) $f_1 = \pi f_2$ and $\phi_1 = \phi_2$
(e) $f_1 = f_2$ and $\phi_1 = \phi_2 + \pi/2$
(f) $f_1 = f_2$ and $\phi_1 = \phi_2 + \pi$

3.2. **(a)** Show that the three functions illustrated in Figure P3.1 are pairwise orthogonal over the interval (–2, 2).
(b) Determine the value of the constant, A, that makes the set of functions in part (a) an orthonormal set.
(c) Express the following waveform, $x(t)$, in terms of the orthonormal set of part (b).

$$x(t) = \begin{cases} 1 & \text{for } 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

3.3. Consider the functions

$$\psi_1(t) = \exp(-|t|) \quad \text{and} \quad \psi_2 = 1 - A\exp(-2|t|)$$

Determine the constant, A, such that $\psi_1(t)$ and $\psi_2(t)$ are orthogonal over the interval $(-\infty, \infty)$.

3.4. Assume that in a binary digital communication system, the signal component out of the correlator receiver is $a_i(T) = +1$ or -1 V with equal probability. If the Gaussian noise at the correlator output has unit variance, find the probability of a bit error.

3.5. A bipolar binary signal, $s_i(t)$, is a +1- or –1-V pulse during the interval $(0, T)$. Additive white Gaussian noise having two-sided power spectral density of 10^{-3} W/Hz is added to the signal. If the received signal is detected with a matched filter, determine the maximum bit rate that can be sent with a bit error probability of $P_B \le 10^{-3}$.

3.6. Bipolar pulse signals, $s_i(t)$ $(i = 1, 2)$, of amplitude ±1 V are received in the presence of AWGN that has a variance of 0.1 V^2. Determine the optimum (minimum probability of error) detection threshold, γ_0, for matched filter detection if the a priori probabilities are: **(a)** $P(s_1) = 0.5$; **(b)** $P(s_1) = 0.7$; **(c)** $P(s_1) = 0.2$. **(d)** Explain the effect of the a priori probabilities on the value of γ_0. {*Hint:* Refer to Equations (B.10) to (B.12).}

3.7. A binary communication system transmits signals $s_i(t)$ $(i = 1, 2)$. The receiver test statistic $z(T) = a_i + n_0$, where the signal component a_i is either $a_1 = +1$ or $a_2 = -1$ and the noise component n_0 is uniformly distributed, yielding the conditional density functions $p(z|s_i)$ given by

$$p(z|s_1) = \begin{cases} \frac{1}{2} & \text{for } -0.2 \le z \le 1.8 \\ 0 & \text{otherwise} \end{cases}$$

and

$$p(z|s_2) = \begin{cases} \frac{1}{2} & \text{for } -1.8 \le z \le 0.2 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability of a bit error, P_B, for the case of equally likely signaling and the use of an optimum decision threshold.

3.8. **(a)** What is the theoretical minimum system bandwidth needed for a 10-Mbits/s signal using 16-level PAM without ISI?

(b) How large can the filter roll-off factor be if the allowable system bandwidth is 1.375 MHz?

3.9. A voice signal (300 to 3300 Hz) is digitized such that the quantization distortion $\le \pm 0.1\%$ of the peak-to-peak signal voltage. Assume a sampling rate of 8000 samples/s and a multilevel PAM waveform with $M = 32$ levels. Find the theoretical minimum system bandwidth that avoids ISI.

3.10. Binary data at 9600 bits/s are transmitted using 8-ary PAM modulation with a system using a raised cosine roll-off filter characteristic. The system has a frequency response out to 2.4 kHz.

(a) What is the symbol rate?

(b) What is the roll-off factor of the filter characteristic?

3.11. A voice signal in the range 300 to 3300 Hz is sampled at 8000 samples/s. We may transmit these samples directly as PAM pulses or we may first convert each sample to a PCM format and use binary (PCM) waveforms for transmission.

(a) What is the minimum system bandwidth required for the detection of PAM with no ISI and with a filter roll-off characteristic of $r = 1$?

(b) Using the same filter roll-off characteristic, what is the minimum bandwidth required for the detection of binary (PCM) waveforms if the samples are quantized to eight levels?

(c) Repeat part (b) using 128 quantization levels.

3.12. An analog signal is PCM formatted and transmitted using binary waveforms over a channel that is bandlimited to 100 kHz. Assume that 32 quantization levels are used and that the overall equivalent transfer function is of the raised cosine type with roll-off $r = 0.6$.

(a) Find the maximum bit rate that can be used by this system without introducing ISI.

(b) Find the maximum bandwidth of the original analog signal that can be accommodated with these parameters.

(c) Repeat parts (a) and (b) for transmission with 8-ary PAM waveforms.

3.13. Assume that equally-likely RZ binary pulses are coherently detected over a Gaussian channel with $N_0 = 10^{-8}$ Watt/Hz. Assume that synchronization is perfect, and that the received pulses have an amplitude of 100 mV. If the bit-error probability specification is $P_B = 10^{-3}$, find the largest data rate that can be transmitted using this system.

3.14. Consider that NRZ binary pulses are transmitted along a cable that attenuates the signal power by 3 dB (from transmitter to receiver). The pulses are coherently detected at the receiver, and the data rate is 56 kbit/s. Assume Gaussian noise with $N_0 = 10^{-6}$ Watt/Hz. What is the minimum amount of power needed at the transmitter in order to maintain a bit-error probability of $P_B = 10^{-3}$?

3.15. Show that the Nyquist minimum bandwidth for a random binary sequence sent with ideal-shaped bipolar pulses is the same as the noise equivalent bandwidth. Hint: the power spectral density for a random bipolar sequence is given in Equation (1.38) and the noise equivalent bandwidth is defined in Section 1.7.2.

3.16. Consider the 4-ary PAM-modulated sequence of message symbols {+1 +1 −1 +3 +1 +3}, where the members of the alphabet set are: {±1, ±3}. The pulses have been shaped with a root-raised cosine filter such that the support time of each filtered pulse is 6-symbol times, and the transmitted sequence is the analog waveform shown in Figure 2.47a. Note that the waveform appears "smeared" due to the filter-induced ISI. Show how a bank of N correlators can be implemented to perform matched-filter demodulation of the received pulse sequence, $r(t)$, where N corresponds to the number of symbols in the pulse-support time. [*Hint:* For the bank of correlators, use reference signals of the form $s_1(t - kT)$, where $k = 0, \ldots, 5$ and T is the symbol time.]

3.17. A desired impulse response of a communication system is the ideal $h(t) = \delta(t)$, where $\delta(t)$ is the impulse function. Assume that the channel introduces ISI so that the overall impulse response becomes $h(t) = \delta(t) + \alpha\delta(t - T)$, where $\alpha < 1$, and T is the symbol time. Derive an expression for the impulse response of a zero-forcing filter that will equalize the effects of ISI. Demonstrate that this filter suppresses the ISI. If the resulting suppression is deemed inadequate, how can the filter design be modified to increase the ISI suppression further?

3.18. The result of a single pulse (impulse) transmission is a received sequence of samples (impulse response), with values 0.1, 0.3, –0.2, 1.0, 0.4, –0.1, 0.1, where the leftmost sample is the earliest. The value 1.0 corresponds to the mainlobe of the pulse, and the other entries correspond to adjacent samples. Design a 3-tap transversal equalizer that forces the ISI to be zero at one sampling point on each side of the mainlobe. Calculate the values of the equalized output pulses at times $k = 0, \pm1, \ldots, \pm3$. After equalization, what is the largest magnitude sample contributing to ISI, and what is the sum of all the ISI magnitudes?

3.19. Repeat problem 3.18 for the case of a channel impulse response described by the following received samples: 0.01, 0.02, –0.03, 0.1, 1.0, 0.2, –0.1, 0.05, 0.02. Use a computer to find the weights of a nine-tap transversal equalizer to meet the minimum MSE criterion. Calculate the values of the equalized output pulses at times $k = 0, \pm1, \ldots, \pm8$. After equalization, what is the largest magnitude sample contributing to ISI, and what is the sum of all the ISI magnitudes?

3.20. In this chapter, it has been emphasized that signal-processing devices, such as multipliers and integrators, typically deal with signals having units of *volts*. Therefore, the transfer functions of such processors must accommodate these units. Draw a block diagram of a product-integrator showing the signal units on each of the wires, and the device transfer functions in each of the blocks (Hint: see Section 3.2.5.1).

QUESTIONS

3.1. In the case of *baseband* signaling, the received waveforms are already in a pulse-like form. Why then, is a demodulator needed to recover the pulse waveform? (See Chapter 3, introduction.)

3.2. Why is E_b/N_0 a natural figure-of-merit for digital communication systems? (See Section 3.1.5.)

3.3. When representing timed events, what dilemma can easily result in confusing the most-significant bit (MSB) and the least-significant bit (LSB)? (See Section 3.2.3.1.)

3.4. The term *matched-filter* is often used synonymously with *correlator.* How is that possible when their mathematical operations are different? (See Section 3.2.3.1.)

3.5. Describe the two fair ways of comparing different curves that depict bit-error probability versus E_b/N_0. (See Section 3.2.5.3.)

3.6. Are there other pulse-shaping filter functions, besides the *raised-cosine,* that exhibit zero ISI? (See Section 3.3.)

3.7. Describe a reasonable goal in endeavoring to *compress bandwidth* to the minimum possible, without incurring ISI. (See Section 3.3.1.1.)

3.8. The error performance of digital signaling suffers primarily from two degradation types: *loss* in signal-to-noise ratio, and *distortion* resulting in an irreducible bit-error probability. How do they differ? (See Section 3.3.2.)

3.9. Often times, providing more E_b/N_0 will not mitigate the degradation due to *intersymbol interference* (ISI). Explain why this is the case. (See Section 3.3.2.)

3.10. Describe the difference between equalizers that use a *zero-forcing* solution, and those that use a *minimum mean-square error* solution? (See Section 3.4.3.1.)

EXERCISES

Using the Companion CD, run the exercises associated with Chapter 3.